

A weighted L_p -theory for parabolic PDEs with BMO coefficients on C^1 -domains

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Abstract

In this paper we present a weighted L_p -theory of second-order parabolic partial differential equations defined on C^1 domains. The leading coefficients are assumed to be measurable in time variable and have VMO (vanishing mean oscillation) or small BMO (bounded mean oscillation) with respect to space variables, and lower order coefficients are allowed to be unbounded and to blow up near the boundary. Our BMO condition is slightly relaxed than the others in the literature.

Keywords: Parabolic equations, Weighted Sobolev spaces, L_p -theory, BMO coefficients, VMO coefficients.

AMS 2000 subject classifications: 35K20, 35R05.

1 Introduction

In this article we are dealing with a weighted L_p -theory of the parabolic equation:

$$\begin{aligned} u_t &= a^{ij}(t, x)u_{x^i x^j} + b^i(t, x)u_{x^i} + c(t, x)u + f, \quad (t, x) \in (0, T) \times \mathcal{O} \\ u(t, x) &= 0, \quad (t, x) \in (0, T) \times \partial\mathcal{O} \quad ; \quad u(0, x) = u_0(x), \quad x \in \mathcal{O}, \end{aligned} \quad (1.1)$$

where indices i and j run from 1 to d with the summation convention on i and j being enforced, and \mathcal{O} is either a half space or a bounded C^1 -domain. It is assumed that leading coefficients a_{ij} are measurable in t and have VMO or small BMO with respect to x , and lower order coefficients b^i and c satisfy

$$\lim_{\rho(x) \rightarrow 0} \sup_t (\rho(x)|b^i(t, x)| + \rho^2(x)|c(t, x)|) = 0, \quad (1.2)$$

where $\rho(x) = \text{dist}(x, \partial\mathcal{O})$. Note that (1.2) is satisfied if, for instance, $|b^i(t, x)| \leq N\rho^{-1+\varepsilon}(x)$ and $|c(t, x)| \leq N\rho^{-2+\varepsilon}(x)$ for some constants $\varepsilon, N > 0$. Also note that b^i and c are allowed to be unbounded and blow up near the boundary.

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We look for solutions in function spaces with weights, in which the derivatives of solutions are allowed to blow up near the boundary. In particular, we prove that if $\alpha \in (-1, p-1)$, $u_0 = 0$ and $\rho f \in L_p((0, T), L_p(\mathcal{O}, \rho^\alpha(x)dx))$, then equation (1.1) has a unique solution u so that $u|_{\partial\mathcal{O}} = 0$, and for this solution we have

$$\int_0^T \int_{\mathcal{O}} (|\rho^{-1}u|^p + |u_x|^p + |\rho u_{xx}|^p) \rho^\alpha(x) dx dt \leq N(p, d, c) \int_0^T \int_{\mathcal{O}} |\rho f|^p \rho^\alpha(x) dx dt. \quad (1.3)$$

The condition $\alpha \in (-1, p-1)$ is sharp even for the heat equation $u_t = \Delta u + f$ (see [21]). Also, unless much stronger condition on the constant α is imposed, in general (1.3) is false even for the heat equation if \mathcal{O} is just a Lipschitz domain (see [10]).

Our motivation of using such weighted Sobolev spaces lies in the L_p -theory of stochastic partial differential equations (SPDEs) of the type

$$dw = (a^{ij}w_{x^i x^j} + b^i w_{x^i} + cw + \tilde{f})dt + (\sigma^{ik}w_{x^i} + g^k)dB_t^k, \quad (1.4)$$

where B_t^k ($k = 1, 2, \dots$) are independent one-dimensional Brownian motions defined on a probability space $(\Omega', \mathcal{F}, P)$, and all the coefficients and inputs \tilde{f}, g^k and the solution w are random functions depending also on (t, x) . It is known that, unless certain compatibility conditions are assumed, the second derivatives $w_{x^i x^j}$ may blow up near the boundary. Hence, we have to measure the second derivatives $w_{x^i x^j}$ using appropriate weights near the boundary. It is not hard to see that our weighted L_p -theory of equation (1.1) with BMO coefficients easily yields the corresponding L_p -theory for SPDE (1.4) with BMO coefficients. Indeed, for simplicity assume $b^i = c = \sigma^{ik} = 0$ and consider the stochastic heat equation

$$dv = (\Delta v + \tilde{f})dt + g^k dB_t^k. \quad (1.5)$$

It is well known (e.g. [11, 21, 25]) that

$$\mathbb{E} \int_0^T \int_{\mathcal{O}} (|\rho^{-1}v|^p + |v_x|^p + |\rho v_{xx}|^p) \rho^\alpha(x) dx dt \leq N \mathbb{E} \int_0^T \int_{\mathcal{O}} (|\rho \tilde{f}|^p + |g|_{\ell_2}^p + |\rho g_x|_{\ell_2}^p) \rho^\alpha(x) dx dt,$$

where $\mathbb{E}X := \int_{\Omega'} X dP$. Obviously for each $\omega \in \Omega'$, $\bar{u} := w - v$ satisfies the deterministic equation

$$\bar{u}_t = a^{ij}\bar{u}_{x^i x^j} + (a^{ij} - \delta^{ij})v_{x^i x^j},$$

and one gets estimates of \bar{u} from (1.3) for each $\omega \in \Omega'$. Since $w = v + \bar{u}$, the weighted L_p norm of $\rho^{-1}w, w_x$ and ρw_{xx} are obtained for free. Therefore inequality (1.3) for the deterministic equation yields an extension of existing L_p -theories (e.g. [11, 18, 21, 25]) of SPDE (1.4) with continuous leading coefficients.

The Sobolev space theory of second-order parabolic and elliptic equations with discontinuous coefficients has been studied extensively in the last few decades. The famous counterexample of Nadirashvili for the solvability of equations with general discontinuous coefficients made people to look for particular type of discontinuity. Among them, VMO condition (or small BMO condition) is

very sharp and important from mathematical point of view. For practical motivation, we mention that the uniqueness result for elliptic equations with discontinuous coefficients has connection to the weak uniqueness of solutions of the corresponding stochastic differential equations.

The study of equations with VMO coefficients was initiated in [4] (elliptic equations) and in [1] (parabolic equations) and continued in, for instance, [1], [2], [3] and [5]. In [17] N.V. Krylov gave a unified approach to investigating the L_p solvability of both divergence and non-divergence form of parabolic and elliptic equations with leading coefficients that are measurable in time variable and have VMO (or small BMO) with respect to spatial variables. Since the publication of [17], the theory kept evolved, especially in the direction of partially VMO coefficients. We refer the reader to e.g. [6], [7] and [9]. The reader can view our article as a weighted version of existing L_p -theories with small BMO (or VMO) coefficients.

Our BMO (or VMO) condition is slightly relaxed than the others in the literature (see Remarks 5.2 and 7.8) because we impose small BMO condition only on the balls away from the boundary, that is balls of the type $B_r(x) \subset \mathcal{O}$ with $r \leq \kappa_0 \rho(x) \wedge \delta$, where $\delta, \kappa_0 \in (0, 1)$ are some constants. Thus no restriction is imposed on the balls intersecting with the boundary. This relaxation has become possible due to the method found in [12]. The key is to establish weighted sharp function estimate (see Lemma 5.4 below) and apply the weighted version of Fefferman-Stein and Hardy-Littlewood theorems developed in [12]. By the way, if a^{ij} are continuous in x , then our results were already introduced in [14, 19]. Our article is a natural extension of [14, 19] to the equations with discontinuous coefficients.

The article is organized as follows. In section 2 we introduce our weighted Sobolev spaces and the weighted version of Fefferman-Stein and Hardy-Littlewood theorems. In section 3 we discuss local estimates which we use later. In Section 4 and 5 we present sharp function estimates and a priori estimates. In Section 6 and 7 we prove our main results using all previous preparations.

We finish the introduction with some notations. As usual \mathbb{R}^d stands for the Euclidean space of points $x = (x^1, \dots, x^d)$, $\mathbb{R}_+^d := \{x = (x^1, \dots, x^d) \in \mathbb{R}^d : x^1 > 0\}$ and $B_r(x) := \{y \in \mathbb{R}^d : |x - y| < r\}$. For $i = 1, \dots, d$, multi-indices $\alpha = (\alpha_1, \dots, \alpha_d)$, $\alpha_i \in \{0, 1, 2, \dots\}$, and functions $u(x)$ we set

$$u_{x^i} = \frac{\partial u}{\partial x^i} = D_i u, \quad D^\alpha u = D_1^{\alpha_1} \cdot \dots \cdot D_d^{\alpha_d} u, \quad |\alpha| = \alpha_1 + \dots + \alpha_d.$$

We also use the notation D^m for a partial derivative of order m with respect to x ; for instance, we use $Du = u_x$ for a first order derivative of u and $D^2u = u_{xx}$ for a second order derivative of u . If we write $N = N(a, b, \dots)$, this means that the constant N depends only on a, b, \dots . $A \sim B$ means $A \leq N_1 B$ and $B \leq N_2 A$ for some constants N_1, N_2 .

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2 Preliminaries: weighted Sobolev spaces on \mathbb{R}_+^d

For any $p > 1$ and $\gamma \in \mathbb{R}$, define the space of Bessel potential $H_p^\gamma = H_p^\gamma(\mathbb{R}^d)$ as the space of all distributions u on \mathbb{R}^d such that

$$\|u\|_{H_p^\gamma} = \|(1 - \Delta)^{\gamma/2} u\|_{L_p} := \|\mathcal{F}^{-1}[(1 + |\xi|^2)^{\gamma/2} \mathcal{F}(u)(\xi)]\|_{L_p} < \infty,$$

where \mathcal{F} is the Fourier transform. Then H_p^γ is a Banach space with the given norm and $C_0^\infty(\mathbb{R}^d)$ is dense in H_p^γ (see [28]). If γ is a nonnegative integer, then H_p^γ is the usual Sobolev space, that is,

$$H_p^\gamma = \{u : D^\alpha u \in L_p, |\alpha| \leq \gamma\}, \quad \|u\|_{H_p^\gamma}^p \sim \sum_{|\alpha| \leq \gamma} \int_{\mathbb{R}^d} |D^\alpha u|^p dx.$$

It is well known that, for any multi-index α , the operator $D^\alpha : H_p^\gamma \rightarrow H_p^{\gamma-|\alpha|}$ is bounded. On the other hand, if $\text{supp } u \subset (a, b) \times \mathbb{R}^{d-1}$, where $-\infty < a < b < \infty$, then (see e.g. Remark 1.13 in [19])

$$\|u\|_{H_p^\gamma} \leq N(d, a, b) \|u_x\|_{H_p^{\gamma-1}}. \quad (2.1)$$

Also recall that if $|\gamma| \leq n$ for some integer n and $|a|_n := \sup_{|\alpha| \leq n} \sup_x |D^\alpha a| < \infty$ then (see e.g. Lemma 5.2 of [18] for a sharper result)

$$\|au\|_{H_p^\gamma} \leq N(d, \gamma) |a|_n \|u\|_{H_p^\gamma}. \quad (2.2)$$

Next we recall definitions and properties of the weighted Sobolev spaces $H_{p,\theta}^\gamma$ introduced in [19] (also see [20, 25, 26]). The particular case $H_{2,d}^\gamma$, i.e. $\theta = d$ and $p = 2$, is introduced in [24]. For $p > 1, \theta \in \mathbb{R}$ and a nonnegative integer n we define

$$H_{p,\theta}^n := \{u : (x^1)^{|\alpha|} D^\alpha u \in L_p(\mathbb{R}_+^d, (x^1)^{\theta-d} dx), |\alpha| \leq n\},$$

that is, $u \in H_{p,\theta}^n$ if and only if

$$\sum_{|\alpha| \leq n} \int_{\mathbb{R}_+^d} |(x^1)^{|\alpha|} D^\alpha u(x)|^p (x^1)^{\theta-d} dx < \infty. \quad (2.3)$$

We remark that the space $H_{p,\theta}^n$ is different from $W^{n,p}(\mathbb{R}_+^d, x^1, \varepsilon)$ introduced in [22], where

$$W^{n,p}(\mathbb{R}_+^d, x^1, \varepsilon) := \{u : D^\alpha u \in L_p(\mathbb{R}_+^d, (x^1)^\varepsilon dx), |\alpha| \leq n\}. \quad (2.4)$$

For general $\gamma \in \mathbb{R}$ we define the spaces $H_{p,\theta}^\gamma$ as follows. Fix a nonnegative function $\zeta(x) = \zeta(x^1) \in C_0^\infty(\mathbb{R}_+)$ such that

$$\sum_{n=-\infty}^{\infty} \zeta^p(e^n x^1) > c > 0, \quad \forall x^1 \in \mathbb{R}_+, \quad (2.5)$$

where c is a constant. Note that any nonnegative function ζ with $\zeta > 0$ on $[1, e]$ satisfies (2.5). For $\theta \in \mathbb{R}, p > 1$ and $\gamma \in \mathbb{R}$, let $H_{p,\theta}^\gamma = H_{p,\theta}^\gamma(\mathbb{R}_+^d)$ denote the set of all distributions u on \mathbb{R}_+^d such that

$$\|u\|_{H_{p,\theta}^\gamma}^p := \sum_{n \in \mathbb{Z}} e^{n\theta} \|\zeta(\cdot) u(e^n \cdot)\|_{H_p^\gamma}^p < \infty. \quad (2.6)$$

It is not hard to show that for different η satisfying (2.5), we get the same spaces $H_{p,\theta}^\gamma$ with equivalent norms. Indeed, let $\eta(x) = \eta(x^1) \in C_0^\infty(\mathbb{R}_+)$, then there exists an integer m so that $\xi(x) := \eta(x)[\sum_{n=-\infty}^\infty \zeta(e^n x)]^{-1} = \eta(x)[\sum_{|n| \leq m} \zeta(e^n x)]^{-1} \in C_0^\infty(\mathbb{R}_+)$. Thus by (2.2),

$$\|u(e^n \cdot) \eta(\cdot)\|_{H_p^\gamma}^p = \|u(e^n \cdot) \xi \sum_{|k| \leq m} \zeta(e^k \cdot)\|_{H_p^\gamma}^p \leq N \sum_{|k| \leq m} \|u(e^n \cdot) \zeta(e^k \cdot)\|_{H_p^\gamma}^p \leq N \sum_{|k| \leq m} \|u(e^{n-k} \cdot) \zeta(\cdot)\|_{H_p^\gamma}^p,$$

and therefore we get

$$\sum_{n=-\infty}^\infty e^{n\theta} \|\eta(\cdot) u(e^n \cdot)\|_{H_p^\gamma}^p \leq N \sum_{n=-\infty}^\infty e^{n\theta} \|\zeta(\cdot) u(e^n \cdot)\|_{H_p^\gamma}^p. \quad (2.7)$$

By the same reason the reverse of (2.7) holds if η satisfies (2.5).

To compare (2.3) and (2.6) when $\gamma = n = 0$, denote $L_{p,\theta} := H_{p,\theta}^0$ and note that

$$\sum_n e^{n\theta} \|\zeta(x^1) u(e^n x)\|_{L_p}^p = \int_{\mathbb{R}_+^d} |u(x)|^p \sum_n e^{n(\theta-d)} \zeta^p(e^{-n} x^1) dx =: \int_{\mathbb{R}_+^d} |u(x)|^p \eta_0(x^1) dx,$$

where $\eta_0(x^1) := \sum_n e^{n(\theta-d)} \zeta^p(e^{-n} x^1)$. Obviously the function $\xi_0(t) := \sum_n e^{(n-t)(\theta-d)} \zeta^p(e^{t-n})$ is bounded 1-periodic function having positive minimum and $\eta_0(x^1) = \xi_0(\ln x^1)(x^1)^{\theta-d}$. It follows that for some $N = N(\zeta) > 0$ we have

$$N^{-1} \|u\|_{L_{p,\theta}}^p \leq \int_{\mathbb{R}_+^d} |u|^p (x^1)^{\theta-d} dx \leq N \|u\|_{L_{p,\theta}}^p.$$

Therefore (2.3) and (2.6) give equivalent norms if $\gamma = n = 0$. Actually, in general if $\gamma = n$ is a nonnegative integer, then (see Corollary 2.3 of [19] for details)

$$\|u\|_{H_{p,\theta}^n}^p \sim \sum_{|\alpha| \leq n} \int_{\mathbb{R}_+^d} |(x^1)^{|\alpha|} D^\alpha u(x)|^p (x^1)^{\theta-d} dx. \quad (2.8)$$

Let M^α be the operator of multiplying by $(x^1)^\alpha$ and $M := M^1$. We write $u \in M^\alpha H_{p,\theta}^\gamma$ if $M^{-\alpha} u \in H_{p,\theta}^\gamma$. For $\nu \in (0, 1]$, denote

$$|u|_C = \sup_{x \in \mathbb{R}_+^d} |u(x)|, \quad [u]_{C^\nu} = \sup_{x \neq y} \frac{|u(x) - u(y)|}{|x - y|^\nu}.$$

Below are other important properties of the spaces $H_{p,\theta}^\gamma$ taken from [19, 20].

Lemma 2.1. *Let $\gamma, \theta \in \mathbb{R}$ and $p \in (1, \infty)$.*

(i) $C_0^\infty(\mathbb{R}_+^d)$ is dense in $H_{p,\theta}^\gamma$.

(ii) Assume that $\gamma = m + \nu + d/p$ for some $m = 0, 1, \dots$ and $\nu \in (0, 1]$. Then for any $u \in H_{p,\theta}^\gamma$ and $i \in \{0, 1, \dots, m\}$, we have

$$|M^{i+\theta/p} D^i u|_C + [M^{m+\nu+\theta/p} D^m u]_{C^\nu} \leq N \|u\|_{H_{p,\theta}^\gamma}. \quad (2.9)$$

(iii) Let $|\gamma| \leq n$ and $|a|_n^{(0)} := \sup_{|\alpha| \leq n} \sup_x M^{|\alpha|} |D^\alpha a| < \infty$, then

$$\|au\|_{H_{p,\theta}^\gamma} \leq N(d, \gamma, \theta) |a|_n^{(0)} \|u\|_{H_{p,\theta}^\gamma}. \quad (2.10)$$

(iv) Let $\alpha \in \mathbb{R}$. Then $M^\alpha H_{p,\theta+\alpha p}^\gamma = H_{p,\theta}^\gamma$ and

$$\|u\|_{H_{p,\theta}^\gamma} \leq N \|M^{-\alpha} u\|_{H_{p,\theta+\alpha p}^\gamma} \leq N \|u\|_{H_{p,\theta}^\gamma}.$$

(v) $MD, DM : H_{p,\theta}^\gamma \rightarrow H_{p,\theta}^{\gamma-1}$ are bounded linear operators, and

$$\|u\|_{H_{p,\theta}^\gamma} \leq N \|u\|_{H_{p,\theta}^{\gamma-1}} + N \|Mu_x\|_{H_{p,\theta}^{\gamma-1}} \leq N \|u\|_{H_{p,\theta}^\gamma},$$

$$\|u\|_{H_{p,\theta}^\gamma} \leq N \|u\|_{H_{p,\theta}^{\gamma-1}} + N \|(Mu)_x\|_{H_{p,\theta}^{\gamma-1}} \leq N \|u\|_{H_{p,\theta}^\gamma}.$$

(vi) If $\theta \neq d-1, d-1+p$, then

$$\|u\|_{H_{p,\theta}^\gamma} \leq N \|Mu_x\|_{H_{p,\theta}^{\gamma-1}}, \quad \|u\|_{H_{p,\theta}^\gamma} \leq N \|(Mu)_x\|_{H_{p,\theta}^{\gamma-1}}. \quad (2.11)$$

(vii) For $i = 0, 1$ let $\kappa \in [0, 1]$, $p_i \in (1, \infty)$, $\gamma_i, \theta_i \in \mathbb{R}$ and assume the relations

$$\gamma = \kappa \gamma_1 + (1 - \kappa) \gamma_0, \quad \frac{1}{p} = \frac{\kappa}{p_1} + \frac{1 - \kappa}{p_0}, \quad \frac{\theta}{p} = \frac{\theta_1 \kappa}{p_1} + \frac{\theta_0 (1 - \kappa)}{p_0}.$$

Then

$$\|u\|_{H_{p,\theta}^\gamma} \leq N \|u\|_{H_{p_1,\theta_1}^{\gamma_1}}^\kappa \|u\|_{H_{p_0,\theta_0}^{\gamma_0}}^{1-\kappa}.$$

Remark 2.2. Let $\theta \in (d-1, d-1+p)$ and n be a nonnegative integer. By Lemma 2.1 (iv), (vi)

$$\|M^{-n} v\|_{H_{p,\theta}^\gamma} \leq N \|D^n v\|_{H_{p,\theta}^{\gamma-n}} \quad (2.12)$$

for any $v \in C_0^\infty(\mathbb{R}_+^d)$. Indeed, since $\theta + mp \neq d-1, d-1+p$ for any integer m

$$\begin{aligned} \|M^{-n} v\|_{H_{p,\theta}^\gamma} &\leq N \|M^{-1} v\|_{H_{p,\theta-(n-1)p}^\gamma} \leq N \|v_x\|_{H_{p,\theta-(n-1)p}^{\gamma-1}} \\ &\leq N \|M^{-1} v_x\|_{H_{p,\theta-(n-2)p}^{\gamma-1}} \leq N \|D^2 v\|_{H_{p,\theta-(n-2)p}^{\gamma-2}} \dots \end{aligned}$$

Next, we introduce Fefferman-Stein and Hardy-Littlewood theorems in weighted L_p -spaces. Denote

$$\Omega := \mathbb{R} \times \mathbb{R}_+^d := \{(t, x) = (t, x^1, x^2, \dots, x^d) : x^1 > 0\}.$$

Fix $\alpha \in (-1, \infty)$ and define the weighted measures

$$\nu(dx) = \nu_\alpha(dx) = (x^1)^\alpha dx, \quad d\mu = \mu_\alpha(dt dx) := \nu_\alpha(dx) dt.$$

Let $B'_r(x')$ denote the open ball in \mathbb{R}^{d-1} of radius r with center x' . For $x = (x^1, x') \in \mathbb{R}_+^d$ and $t \in \mathbb{R}$, denote

$$\mathcal{B}_r(x) = \mathcal{B}_r(x^1, x') = (x^1 - r, x^1 + r) \times B'_r(x'), \quad \mathcal{Q}_r(t, x) := (t, t + r^2) \times \mathcal{B}_r(x).$$

By \mathbb{Q} we mean the collection of all such open sets $\mathcal{Q}_r(t, x) \subset \Omega$. For $f \in L_{1,loc}(\Omega, \mu)$ we define

$$f_{\mathcal{Q}} = \int_{\mathcal{Q}} f d\mu, \quad \mathbb{M}f(t, x) = \sup_{\mathcal{Q}} \int_{\mathcal{Q}} f d\mu, \quad (f)^{\sharp}(t, x) = \sup_{\mathcal{Q}} \int_{\mathcal{Q}} |f - f_{\mathcal{Q}}| d\mu,$$

where the supremum is taken for all $\mathcal{Q} \in \mathbb{Q}$ containing (t, x) .

Theorem 2.3. ([12]) (Fefferman-Stein) Let $p \in (1, \infty)$. Then for any $f \in L_p(\Omega, \mu)$, we have

$$\|f\|_{L_p(\Omega, \mu)} \leq N \|f^{\sharp}\|_{L_p(\Omega, \mu)},$$

where $N = N(\alpha, p, d)$.

Theorem 2.4. ([12]) (Hardy-Littlewood) Let $p \in (1, \infty)$. Then for $f \in L_p(\Omega, \mu)$ we have

$$\|\mathbb{M}f\|_{L_p(\Omega, \mu)} \leq N \|f\|_{L_p(\Omega, \mu)},$$

where $N = N(\alpha, p, d)$.

3 Some local estimates of solutions

In this section we develop some local estimates of $D^{\beta}u$ for any multi-index β , where u is a solution of the equation:

$$u_t + a^{ij}u_{x^i x^j} = f, \quad (t, x) \in \Omega := \mathbb{R} \times \mathbb{R}_+^d. \quad (3.1)$$

In particular, we prove that if $f = 0$ in $\mathcal{Q}_r(r) := (0, r^2) \times (0, 2r) \times B'_r(0)$ then for any $s \in (0, r)$ and $\theta \in (d-1, d-1+p)$,

$$\max_{(t, x) \in \mathcal{Q}_s(s)} (|D^{\beta}u_{xx}|^p + |D^{\beta}u_t|^p) \leq N(r, s, \beta, \theta) \int_{\mathcal{Q}_r(r)} |u|^p (x^1)^{\theta-d+p} dx dt.$$

The estimates obtained here will be used to estimate the sharp function of u_{xx} in the next section.

Throughout this section we assume the following.

Assumption 3.1. $a^{ij} = a^{ij}(t)$ are independent of x , and there exist constants $\delta, K > 0$ so that

$$\delta |\xi|^2 \leq a^{ij}(t) \xi^i \xi^j \leq K |\xi|^2, \quad \forall \xi \in \mathbb{R}^d. \quad (3.2)$$

For $-\infty \leq S < T \leq \infty$, we define the Banach spaces

$$\mathbb{H}_{p, \theta}^{\gamma}(S, T) := L_p((S, T), H_{p, \theta}^{\gamma}), \quad \mathbb{H}_{p, \theta}^{\gamma}(T) := \mathbb{H}_{p, \theta}^{\gamma}(0, T), \quad \mathbb{L}_{p, \theta}(S, T) := H_{p, \theta}^0(S, T), \quad \mathbb{L}_{p, \theta}(T) := \mathbb{L}_{p, \theta}(0, T)$$

with the norms given by

$$\|u\|_{\mathbb{H}_{p, \theta}^{\gamma}(S, T)} = \left[\int_S^T \|u(t)\|_{H_{p, \theta}^{\gamma}}^p dt \right]^{1/p}.$$

Finally, we set $U_{p, \theta}^{\gamma} := M^{1-2/p} H_{p, \theta}^{\gamma-2/p}$ with the norm

$$\|u\|_{U_{p, \theta}^{\gamma}} := \|M^{-1+2/p} u\|_{H_{p, \theta}^{\gamma-2/p}}.$$

First we recall a Krylov's result for equations with coefficients independent of x .

Lemma 3.2. Let $d - 1 < \theta < d - 1 + p$, $p \in (1, \infty)$, $\gamma \in \mathbb{R}$ and $T \in (0, \infty]$. Then for any $f \in M^{-1}\mathbb{H}_{p,\theta}^\gamma(T)$ and $u_0 \in U_{p,\theta}^{\gamma+2}$, the equation

$$u_t = a^{ij}u_{x^i x^j} + f, \quad u(0) = u_0 \quad (3.3)$$

has a unique solution (in the sense of distributions, see Remark 3.3 below) u in $M\mathbb{H}_{p,\theta}^{\gamma+2}(T)$, and for this solution

$$\|M^{-1}u\|_{\mathbb{H}_{p,\theta}^{\gamma+2}(T)} \leq N \left(\|Mf\|_{\mathbb{H}_{p,\theta}^\gamma(T)} + \|u_0\|_{U_{p,\theta}^{\gamma+2}} \right), \quad (3.4)$$

where $N = N(\delta, K, \theta, \gamma, p)$.

Proof. See Theorem 5.6 of [19]. □

For any distribution h on \mathbb{R}_+^d and $\phi \in C_0^\infty(\mathbb{R}_+^d)$, by (h, ϕ) we denote the image of ϕ under h .

Remark 3.3. We say that u is a solution of (3.3) in the sense of distributions if for any $\phi \in C_0^\infty(\mathbb{R}_+^d)$

$$(u(t), \phi) = (u_0, \phi) + \int_0^t (a^{ij}u_{x^i x^j} + f, \phi) ds, \quad \forall t \leq T.,$$

Corollary 3.4. Let $-\infty \leq S < T < \infty$. For any $f \in M^{-1}\mathbb{H}_{p,\theta}^\gamma(S, T)$ and $u_0 \in U_{p,\theta}^{\gamma+2}$, the equation

$$u_t + a^{ij}u_{x^i x^j} = f, \quad t \in (S, T) \quad (3.5)$$

with $u(T) = u_0$ has a unique solution u in $M\mathbb{H}_{p,\theta}^{\gamma+2}(S, T)$, and for this solution

$$\|M^{-1}u\|_{\mathbb{H}_{p,\theta}^{\gamma+2}(S, T)} \leq N \left(\|Mf\|_{\mathbb{H}_{p,\theta}^\gamma(S, T)} + \|u_0\|_{U_{p,\theta}^{\gamma+2}} \right), \quad (3.6)$$

where $N = N(\delta, K, \theta, \gamma, p)$.

Proof. It is enough to consider the time change $t \rightarrow -(t - T)$ and use Lemma 3.2. □

Denote

$$\mathcal{Q}_r(a) = (0, r^2) \times (a - r, a + r) \times B'_r(0), \quad U_r = (-r^2, r^2) \times (-2r, 2r) \times B'_r(0).$$

Lemma 3.5. Let $d - 1 < \theta < d - 1 + p$, $0 < s < r < \infty$, $u(t, x) \in C_0^\infty(\mathbb{R} \times \mathbb{R}_+^d)$ and

$$u_t + a^{ij}(t)u_{x^i x^j} = 0 \quad \text{for } (t, x) \in \mathcal{Q}_r(r).$$

Then for any multi-index $\beta = (\beta^1, \dots, \beta^d)$, we have

$$\begin{aligned} & \int_{\mathcal{Q}_s(s)} (|M^{-1}D^\beta u|^p + |D^\beta u_x|^p + |MD^\beta u_{xx}|^p) (x^1)^{\theta-d} dx dt \\ & \leq N(1+r)^{|\beta|p} \cdot (1+(r-s)^{-2})^{(|\beta|+1)p} \int_{\mathcal{Q}_r(r)} |Mu(t, x)|^p (x^1)^{\theta-d} dx dt, \end{aligned} \quad (3.7)$$

where $N = N(\theta, p, |\beta|, \delta, K)$.

Proof. We use the induction on $|\beta|$.

First, let $|\beta| = 0$. We modify the proof of Lemma 2.4.4 of [16]. Denote $r_0 = s$ and $r_m = s + (r - s) \sum_{j=1}^m 2^{-j}$ for $m = 1, 2, \dots$. Choose smooth functions $\zeta_m \in C_0^\infty(\mathbb{R}^{d+1})$ so that $0 \leq \zeta_m \leq 1$,

$$\zeta_m = 1 \quad \text{on} \quad U_{r_m}, \quad \zeta_m = 0 \quad \text{on} \quad \mathbb{R}^{d+1} \setminus U_{r_{m+1}},$$

$$|\zeta_{mx}| \leq N(r - s)^{-1} 2^m, \quad |\zeta_{mxx}| \leq N(r - s)^{-2} 2^{2m}, \quad |\zeta_{mt}| \leq N(r - s)^{-2} 2^{2m}. \quad (3.8)$$

Note that for each m , $(u\zeta_m)(r^2, x) = 0$ and $u\zeta_m$ satisfies

$$(u\zeta_m)_t + a^{ij}(u\zeta_m)_{x^i x^j} = f_m := \zeta_{mt}u + a^{ij}u\zeta_{mx^i x^j} + 2a^{ij}(u\zeta_{m+1})_{x^i} \zeta_{mx^j}, \quad (t, x) \in (0, r^2) \times \mathbb{R}_+^d.$$

By Corollary 3.4 for $\gamma = 0$,

$$A_m := \|M^{-1}u\zeta_m\|_{\mathbb{H}_{p,\theta}^2(r^2)} \leq N\|Mf_m\|_{\mathbb{L}_{p,\theta}(r^2)}.$$

Denote $B := (\int_{\mathcal{Q}_r(r)} |Mu|^p(x^1)^{\theta-d} dx dt)^{1/p}$. Then by (3.8) and Lemma 2.1,

$$\begin{aligned} \|\zeta_{mt}Mu + a^{ij}Mu\zeta_{mx^i x^j}\|_{\mathbb{L}_{p,\theta}(r^2)} &\leq N(r - s)^{-2} 2^{2m} B, \\ \|a\zeta_{mx}M(u\zeta_{m+1})_x\|_{\mathbb{L}_{p,\theta}(r^2)} &\leq N(r - s)^{-1} 2^m \|M(u\zeta_{m+1})_x\|_{\mathbb{L}_{p,\theta}(r^2)} \leq N(r - s)^{-1} 2^m \|u\zeta_{m+1}\|_{\mathbb{H}_{p,\theta}^1(r^2)}. \end{aligned}$$

By Lemma 2.1 (vii) (take $p_0 = p_1 = p$, $\gamma = 1$, $\gamma_0 = 0$, $\gamma_1 = 2$, $\theta_0 = \theta + p$, $\theta_1 = \theta - p$ and $\kappa = 1/2$) for any $\varepsilon > 0$

$$(r - s)^{-1} 2^m \|u\zeta_{m+1}\|_{\mathbb{H}_{p,\theta}^1(r^2)} \leq \varepsilon A_{m+1} + \varepsilon^{-1} (r - s)^{-2} 2^{2m} B.$$

It follows that (with ε different from the one above),

$$A_m \leq \varepsilon A_{m+1} + N(1 + \varepsilon^{-1})(r - s)^{-2} 2^{2m} B.$$

We take $\varepsilon = \frac{1}{16}$ and get

$$\varepsilon^m A_m \leq \varepsilon^{m+1} A_{m+1} + N\varepsilon^m (1 + \varepsilon^{-1}) 2^{2m} (r - s)^{-2} B,$$

$$A_0 + \sum_{m=1}^{\infty} \varepsilon^m A_m \leq \sum_{m=1}^{\infty} \varepsilon^m A_m + N(r - s)^{-2} B.$$

Note that the series $\sum_{m=1}^{\infty} \varepsilon^m A_m$ converges because $A_m \leq N2^{2m} \|M^{-1}u\|_{\mathbb{H}_{p,d}^2(r^2)}$. By Lemma 2.1(v) and (vi), for any $M^{-1}w \in H_{p,\theta}^2$,

$$\|M^{-1}w\|_{H_{p,\theta}^2} \sim (\|M^{-1}w\|_{L_{p,\theta}} + \|w_x\|_{L_{p,\theta}} + \|Mw_{xx}\|_{L_{p,\theta}}). \quad (3.9)$$

Therefore,

$$\int_{\mathcal{Q}_s(s)} (|M^{-1}u|^p + |u_x|^p + |Mu_{xx}|^p) (x^1)^{\theta-d} dx dt \leq N A_0^p \leq N(r - s)^{-2p} B^p,$$

and (3.7) is proved for $|\beta| = 0$.

Next, assume that (3.7) holds whenever $s < r$ and $|\beta'| = k$, that is

$$\begin{aligned} & \int_{\mathcal{Q}_s(s)} \left(|M^{-1}D^{\beta'}u|^p + |D^{\beta'}u_x|^p + |MD^{\beta'}u_{xx}|^p \right) (x^1)^{\theta-d} dxdt \\ & \leq N(1+r)^{kp} \cdot (1+(r-s)^{-2})^{(k+1)p} \int_{\mathcal{Q}_r(r)} |Mu(t,x)|^p (x^1)^{\theta-d} dxdt. \end{aligned} \quad (3.10)$$

Let $|\beta| = k+1$ and $D^\beta = D_i D^{\beta'}$ for some i and β' with $|\beta'| = k$. Fix a smooth function η so that $\eta = 1$ on U_s , $\eta = 0$ on $\mathbb{R}^{d+1} \setminus U_{(r+s)/2}$, $|\eta_x| \leq N(r-s)^{-1}$, $|\eta_{xx}| \leq N(r-s)^{-2}$ and $|\eta_t| \leq N(r-s)^{-2}$. Note that $v := \eta D^\beta u$ satisfies $v(r^2, \cdot) = 0$ and

$$v_t + a^{ij} v_{x^i x^j} = f := \eta_t D^\beta u + 2a^{ij} \eta_{x^i} D^\beta u_{x^j} + a^{ij} \eta_{x^i x^j} D^\beta u, \quad (t, x) \in (0, r^2) \times \mathbb{R}_+^d.$$

By Corollary 3.4 for $\gamma = 0$ (also note that $x^1 \leq r$ on the support of η and $(r-s)^{-1} \leq 1 + (r-s)^{-2}$),

$$\begin{aligned} \|M^{-1}v\|_{\mathbb{H}_{p,\theta}^2(r^2)}^p & \leq N \|M\eta_t D^\beta u + 2a\eta_x MD^\beta u_x + Ma\eta_{xx} D^\beta u\|_{\mathbb{L}_{p,\theta}(r^2)}^p \\ & \leq N(1+r)^p (1+(r-s)^{-2})^p \int_{\mathcal{Q}_{(s+r)/2}((s+r)/2)} (|D^\beta u|^p + |MD^\beta u_x|^p) \mu(dt dx) \\ & \leq N(1+r)^p (1+(r-s)^{-2})^p \int_{\mathcal{Q}_{(s+r)/2}((s+r)/2)} (|D^{\beta'} u_x|^p + |MD^{\beta'} u_{xx}|^p) \mu(dt dx). \end{aligned}$$

This, (3.9) and (3.10) show that the induction goes through, and hence the lemma is proved. \square

The following result can be found e.g. in [12], and we give a outline of the proof for the sake of the completeness.

Lemma 3.6. *Let $u(t, x) \in C_0^\infty(\mathbb{R} \times \mathbb{R}_+^d)$. Then for any $T > 0$, $p > 1$ and $n = 0, 1, 2, \dots$,*

$$\sup_{t \in [0, T]} \|u(t, \cdot)\|_{H_{p,\theta}^n} \leq N(\|u\|_{\mathbb{H}_{p,\theta}^n(T)} + \|u_t\|_{\mathbb{H}_{p,\theta}^n(T)}).$$

Proof. First of all, it is easy to check that for any $\phi = \phi(t) \in W_p^1((0, T))$ (see [16], p.32)

$$\sup_t |\phi(t)|^p \leq N \int_0^t (|\phi|^p + |\phi'(t)|^p) dt.$$

Thus it suffices to prove

$$\phi(t) := \|u(t, \cdot)\|_{H_{p,\theta}^n} \in W_p^1((0, T)), \quad |\phi'(t)| \leq \|u_t(t, \cdot)\|_{H_{p,\theta}^n}. \quad (3.11)$$

One can prove (3.11) by repeating the proof of Exercise 2.4.8 of [16] (see p.71). It is enough to replace H_p^n there by $H_{p,\theta}^n$. \square

By $C_{loc}^\infty(\Omega)$ we denote the set of real-valued functions u defined on Ω and such that $\zeta u \in C_0^\infty(\Omega)$ for any $\zeta \in C_0^\infty(\Omega)$.

Lemma 3.7. *Let $\theta \in (d-1, d-1+p)$, $s \in (0, r)$ and $u \in C_{loc}^\infty(\Omega)$ satisfies $u_t + a^{ij}(t)u_{x^i x^j} = 0$ for $(t, x) \in \mathcal{Q}_r(r)$. Then for any multi-index $\beta = (\beta^1, \beta^2, \dots, \beta^d)$,*

$$\max_{(t,x) \in \mathcal{Q}_s(s)} (|D^\beta u_{xx}|^p + |D^\beta u_t|^p) \leq N \int_{\mathcal{Q}_r(r)} |u|^p (x^1)^{\theta-d+p} dxdt,$$

where $N = N(\theta, s, r, \beta, p, \delta, K)$.

Proof. Choose the smallest integer n so that $np > (\theta \vee d)$. Note that if $v \in C_0^\infty(\mathbb{R}_+^d)$ and $v(x) = 0$ for $x^1 \geq r$, then by Lemma 2.1(ii) with $\gamma = n$, $i = 0$ and $u = M^{-n}v$,

$$\sup_x |v(x)| \leq N(r) \sup_x |M^{\theta/p} M^{-n} v(x)| \leq N \|M^{-n} v\|_{H_{p,\theta}^n} \leq N(r, p, n) \|D^n v\|_{L_{p,\theta}}, \quad (3.12)$$

where for the last inequality we use Remark 2.2.

Fix $\kappa \in (s, r)$. Let ψ be a smooth function so that $\psi(t, x) = 1$ for $(t, x) \in \mathcal{Q}_s(s)$ and $\psi = 0$ for $(t, x) \notin U_\kappa$. Then $\psi_x = \psi_t = 0$ on $\mathcal{Q}_s(s)$. It follows from (3.12), Lemma 3.6 and Lemma 3.5 that

$$\begin{aligned} \max_{\mathcal{Q}_s(s)} (|D^\beta u_{xx}|^p + |D^\beta u_t|^p) &\leq N \max_{(t,x) \in \mathcal{Q}_s(s)} |(D^\beta \psi u)_{xx}|^p \\ &\leq N \max_{t \in [0, s^2]} \|D_x^n (D^\beta \psi u)_{xx}\|_{L_{p,\theta}}^p \\ &\leq N \left(\|D_x^n (D^\beta \psi u)_{xx}\|_{\mathbb{L}_{p,\theta}(s^2)}^p + \|D_x^n (D^\beta \psi u_t)_{xx}\|_{\mathbb{L}_{p,\theta}(s^2)}^p \right) \\ &\leq N \sum_{|\alpha| \leq n+|\beta|+4} \int_{Q_\kappa(\kappa)} |D^\alpha u|^p (x^1)^{\theta-d} dx dt \\ &\leq N \int_{\mathcal{Q}_r(r)} |Mu|^p (x^1)^{\theta-d} dx dt. \end{aligned}$$

The lemma is proved. \square

4 Sharp function estimates for equations with coefficients independent of x

In this section we introduce some results developed in [12] with detailed proofs for the sake of completeness, and extend Theorem 4.5 and Theorem 4.6 to wider range of weights. These theorems are proved in [12] only for $\theta \in (d-1, d]$ and we extend them for any $\theta \in (d-1, d-1+p)$.

Denote $\nu_\alpha^1(dx^1) = (x^1)^\alpha dx^1$. Recall that

$$\nu_\alpha(dx) = (x^1)^\alpha dx^1 dx' = \nu_\alpha^1(dx^1) dx', \quad \mathcal{B}_r(a) = (a-r, a+r) \times B'_r(0).$$

We start with a weighted Poincaré's inequality.

Lemma 4.1. ([12]) *Let $\alpha > 0$, $p \in [1, \infty)$, $\mathcal{B}_r(a) \subset \mathbb{R}_+^d$, and $u \in C_{loc}^\infty(\mathbb{R}_+^d)$. Then*

$$\int_{\mathcal{B}_r(a)} \int_{\mathcal{B}_r(a)} |u(x) - u(y)|^p \nu_\alpha(dx) \nu_\alpha(dy) \leq 2^{\alpha+1} (2r)^p \nu_\alpha(\mathcal{B}_r(a)) \int_{\mathcal{B}_r(a)} |u_x(x)|^p \nu_\alpha(dx). \quad (4.1)$$

Proof. For $x, y \in \mathcal{B}_r(a)$ we have

$$|u(x) - u(y)|^p \leq (2r)^p \int_0^1 |u_x(tx + (1-t)y)|^p dt$$

and the left-hand side of (4.1) is less than or equal to

$$(2r)^p \int_0^1 I(t) dt = 2(2r)^p \int_{1/2}^1 I(t) dt,$$

where

$$I(t) := \int_{\mathcal{B}_r(a)} \int_{\mathcal{B}_r(a)} |u_x(tx + (1-t)y)|^p \nu_\alpha(dx) \nu_\alpha(dy)$$

and I satisfies $I(t) = I(1-t)$. For each $t \in [1/2, 1]$ and y , $t\mathcal{B}_r(a) + (1-t)y := \{tz + (1-t)y : z \in \mathcal{B}_r(a)\} \subset \mathcal{B}_r(a)$. Substituting $w = tx + (1-t)y$ and noticing $x^1 = (w^1 - (1-t)y^1)/t \leq w^1/t$ since $y^1 \geq 0$, we get

$$\begin{aligned} I(t) &\leq t^{-\alpha-1} \int_{\mathcal{B}_r(a)} \left(\int_{t\mathcal{B}_r(a) + (1-t)y} |u_x(w)|^p \nu_\alpha(dw) \right) \nu_\alpha(dy) \\ &\leq 2^{\alpha+1} \int_{\mathcal{B}_r(a)} \left(\int_{\mathcal{B}_r(a)} |u_x(x)|^p \nu_\alpha(dx) \right) \nu_\alpha(dy) \\ &= 2^{\alpha+1} \nu_\alpha(\mathcal{B}_r(a)) \int_{\mathcal{B}_r(a)} |u_x(x)|^p \nu_\alpha(dx). \end{aligned}$$

Now, (4.1) follows. \square

Lemma 4.2. ([12]) Let $\alpha > 0$. Denote $\nu_\alpha^1(dx^1) = (x^1)^\alpha dx^1$. For any $\mathcal{B}_r^1(a) := (a-r, a+r) \subset \mathbb{R}_+$ we have a non-negative function $\zeta \in C_0^\infty(\mathbb{R}_+)$ and a constant $N = N(\alpha)$ such that

$$\text{supp}(\zeta) \in \mathcal{B}_r^1(a), \quad \int_{\mathcal{B}_r^1(a)} \zeta(x^1) \nu_\alpha^1(dx^1) = 1, \quad (4.2)$$

$$\sup_x \zeta \cdot \nu_\alpha^1(\mathcal{B}_r^1(a)) \leq N, \quad \sup_x |\zeta_{x^1}| \cdot \nu_\alpha^1(\mathcal{B}_r^1(a)) \leq \frac{N}{r}. \quad (4.3)$$

Proof. Choose a nonnegative smooth function $\psi = \psi(x^1) \in C_0^\infty(\mathcal{B}_{1/2}^1(0))$ so that $\int_{\mathbb{R}} \psi(x^1) dx^1 = 1$ and $\psi(x^1) = 0$ for $|x^1| \geq 1/2$. Define

$$\zeta(x^1) = \frac{(x^1)^{-\alpha}}{r} \psi\left(\frac{x^1 - a}{r}\right).$$

Then (4.2) is obvious. Since $r \leq a$ and $(a+r)^{\alpha+1} - (a-r)^{\alpha+1} \leq 2r(\alpha+1)(2a)^\alpha$,

$$\begin{aligned} \sup |\zeta| \cdot \nu_\alpha^1(\mathcal{B}_r^1(a)) &\leq N \sup_{|x^1-a| \leq r/2} \frac{(x^1)^{-\alpha}}{r} \cdot ((a+r)^{\alpha+1} - (a-r)^{\alpha+1}) \\ &\leq N \frac{(a/2)^{-\alpha}}{r} \cdot ((a+r)^{\alpha+1} - (a-r)^{\alpha+1}) \leq N. \end{aligned}$$

Similarly, the last inequality also holds because

$$\begin{aligned} \sup |\zeta_{x^1}| \cdot \nu_\alpha^1(\mathcal{B}_r^1(a)) &\leq N \sup_{|x^1-a| \leq r/2} \left(\frac{(x^1)^{-\alpha}}{r^2} + \frac{(x^1)^{-\alpha-1}}{r} \right) \cdot ((a+r)^{\alpha+1} - (a-r)^{\alpha+1}) \\ &\leq \frac{N}{r} \left(1 + \frac{(2a)^{\alpha+1}}{(a/2)^{\alpha+1}} \right) \leq \frac{N}{r}. \end{aligned}$$

The lemma is proved. \square

Recall that for $t \in \mathbb{R}$, $a \in \mathbb{R}_+$ and $x' \in \mathbb{R}^{d-1}$

$$\mathcal{Q}_r(t, a, x') := (t, t + r^2) \times (a - r, a + r) \times B'_r(x'), \quad \mathcal{Q}_r(a) := \mathcal{Q}_r(0, a, 0).$$

From this point on we fix $\alpha := \theta - d + p$ and denote

$$\begin{aligned} \nu^1(dx^1) &:= (x^1)^\alpha dx^1, \quad \mu(dt dx) = \nu(dx) dt := (x^1)^\alpha dx dt, \\ u_{\mathcal{Q}_r(a)} &= \frac{1}{\mu(\mathcal{Q}_r(a))} \int_{\mathcal{Q}_r(a)} u(t, x) \mu(dx dt). \end{aligned}$$

Lemma 4.3. ([12]) Let $p \in [1, \infty)$, $f^i, g \in C_{loc}^\infty(\Omega)$. Assume that $u \in C_{loc}^\infty(\Omega)$ satisfies the equation

$$u_t + a^{ij} u_{x^i x^j} = f_{x^i}^i + g \quad (4.4)$$

on $\mathcal{Q}_r(a) \subset \Omega$. Then

$$\int_{\mathcal{Q}_r(a)} |u(t, x) - u_{\mathcal{Q}_r(a)}|^p \mu(dt dx) \leq N r^p \int_{\mathcal{Q}_r(a)} (|u_x(t, x)|^p + |f(t, x)|^p + r^p |g(t, x)|^p) \mu(dt dx), \quad (4.5)$$

where $N = N(\theta, p, d, \delta, K)$.

Proof. We follow the outline for the proof of Theorem 4.2.1 in [16]. We take the function ζ corresponding to $\mathcal{B}_r^1(a)$ and $\alpha(= \theta - d + p)$ from Lemma 4.2, and take a nonnegative function $\phi = \phi(x') \in C_0^\infty(B'_1(0))$ with unit integral. Denote $\eta(x') = r^{-d+1} \phi(x'/r)$, $\mathcal{B}_r(a) = (a - r, a + r) \times B'_r(0)$ as before, and for $t \in (0, r^2)$ set

$$\bar{u}(t) := \int_{\mathcal{B}_r(a)} \zeta(y^1) \eta(y') u(t, y) \nu(dy).$$

Then by Jensen's inequality and Poincaré's inequality (Lemma 4.1),

$$\begin{aligned} & \int_{\mathcal{B}_r(a)} |u(t, x) - \bar{u}(t)|^p \nu(dx) \\ &= \int_{\mathcal{B}_r(a)} \left| \int_{\mathcal{B}_r(a)} (u(t, x) - u(t, y)) \zeta(y^1) \eta(y') \nu(dy) \right|^p \nu(dx) \\ &\leq \int_{\mathcal{B}_r(a)} \left(\int_{\mathcal{B}_r(a)} |u(t, x) - u(t, y)|^p \zeta(y^1) \eta(y') \nu(dy) \right) \nu(dx) \\ &\leq |\sup \zeta| \cdot |\sup \eta| \int_{\mathcal{B}_r(a)} \int_{\mathcal{B}_r(a)} |u(t, x) - u(t, y)|^p \nu(dx) \nu(dy) \\ &\leq N r^{-d+1} |\sup \zeta| \cdot \nu(\mathcal{B}_r(a)) r^p \int_{\mathcal{B}_r(a)} |u_x(x)|^p \nu(dx) \\ &\leq N r^{-d+1} |\sup \zeta| \cdot \nu^1((a - r, a + r)) r^{d-1} r^p \int_{\mathcal{B}_r(a)} |u_x(x)|^p \nu(dx) \\ &\leq N r^p \int_{\mathcal{B}_r(a)} |u_x(x)|^p \nu(dx). \end{aligned} \quad (4.6)$$

We observe that for any constant vector $c \in \mathbb{R}$ the left-hand side of (4.5) is less than $2 \cdot 2^p$ times

$$\int_{\mathcal{Q}_r(a)} |u(t, x) - c|^p \mu(dt dx) \leq 2^p \int_{\mathcal{Q}_r(a)} |u(t, x) - \bar{u}(t)|^p \mu(dt dx) + 2^p \nu(\mathcal{B}_r(a)) \int_0^{r^2} |\bar{u}(t) - c|^p dt. \quad (4.7)$$

By (4.6) the first term of the right side of (4.7) is less than (4.5). To estimate the second term, we take $c = \frac{1}{r^2} \int_0^{r^2} \bar{u}(t) dt$. Then by Poincaré's inequality without a weight in variable t we have

$$\nu(\mathcal{B}_r(a)) \int_0^{r^2} |\bar{u}(t) - c|^p dt \leq N \nu(\mathcal{B}_r(a)) (r^2)^p \int_0^{r^2} \left| \int_{\mathcal{B}_r(a)} \zeta(x^1) \eta(x') u_t(t, x) \nu(dx) \right|^p dt. \quad (4.8)$$

To estimate the right side of (4.8), we recall $u_t = -a^{ij}(t) u_{x^i x^j} + f_{x^i}^i + g$. First, to handle the integral with g , we use Jensen's inequality, take the supremum out of the integral to get

$$\begin{aligned} & \nu(\mathcal{B}_r(a)) r^{2p} \int_0^{r^2} \left| \int_{\mathcal{B}_r(a)} \zeta(x^1) \eta(x') g(t, x) \nu(dx) \right|^p dt \\ & \leq \nu(\mathcal{B}_r(a)) r^{2p} |\sup \zeta| |\sup \eta| \int_0^{r^2} \int_{\mathcal{B}_r(a)} |g(t, x)|^p \nu(dx) dt \\ & \leq N \nu^1((a-r, a+r)) r^{d-1} r^{2p} |\sup \zeta| r^{-d+1} \int_0^{r^2} \int_{\mathcal{B}_r(a)} |g(t, x)|^p \nu(dx) dt \\ & \leq N(\theta, p, d) r^{2p} \int_{\mathcal{Q}_r(a)} |g(t, x)|^p \mu(dt dx), \end{aligned}$$

where we used $|\sup \zeta| \nu^1((a-r, a+r)) \leq N$ (see Lemma 4.2).

Next, we handle the integral with $-a^{ij} u_{x^i x^j}$. Fix i, j . Firstly, assume either i or j is 1; say $j = 1$. We use integration by parts and observe

$$\begin{aligned} & \nu(\mathcal{B}_r(a)) (r^2)^p \int_0^{r^2} \left| \int_{\mathcal{B}_r(a)} \zeta(x^1) \eta(x') a^{ij}(t) u_{x^i x^j}(t, x) \nu(dx) \right|^p dt \\ & \leq \nu(\mathcal{B}_r(a)) r^{2p} \int_0^{r^2} \left| \int_{\mathcal{B}_r(a)} \zeta_{x^1}(x^1) \eta(x') a^{ij}(t) u_{x^i}(t, x) \nu(dx) \right|^p dt \\ & \quad + \nu(\mathcal{B}_r(a)) r^{2p} (\alpha - 1)^p \int_0^{r^2} \left| \int_{\mathcal{B}_r(a)} \frac{1}{x^1} \zeta(x^1) \eta(x') a^{ij}(t) u_{x^i}(t, x) \nu(dx) \right|^p dt \\ & =: I_1 + I_2. \end{aligned}$$

For I_2 we use the fact $|a^{ij} u_{x^i}| \leq |a^{ij}| |u_{x^i}| \leq K |u_x|$ and $1/x^1 \leq 2/r$ on the support of ζ . The argument handling the case of g easily shows

$$I_2 \leq N(K, \theta, p, d) r^p \int_{\mathcal{Q}_r(a)} |u_x(t, x)|^p \mu(dt dx).$$

For I_1 we use Hölder's inequality and get

$$\begin{aligned} \nu(\mathcal{B}_r(a)) \cdot \left| \int_{\mathcal{B}_r(a)} \zeta_{x^1} \eta a^{ij} u_{x^i} d\nu \right|^p & \leq \nu(\mathcal{B}_r(a))^p \int_{\mathcal{B}_r(a)} |\zeta_{x^1} \eta a^{ij} u_{x^i}|^p d\nu \\ & \leq N(\nu^1((a-r, a+r))^p r^{(d-1)p} \cdot |\sup \zeta_{x^1}|^p r^{(-d+1)p} \int_{\mathcal{B}_r(a)} |u_x|^p \nu(dx). \end{aligned}$$

Since $\nu^1((a-r, a+r)) \cdot |\sup \zeta_x| \leq N/r$, it easily follows that

$$I_1 \leq N(K, \theta, p, d) r^p \int_{\mathcal{Q}_r(a)} |u_x(t, x)|^p \mu(dt dx).$$

Secondly, if $i, j \neq 1$, by integration by parts, Hölder's inequality and the inequality $\sup |\eta_{x'}| \leq Nr^{-d}$,

$$\begin{aligned}
& \nu(\mathcal{B}_r(a)) r^{2p} \int_0^{r^2} \left| \int_{\mathcal{B}_r(a)} \zeta(x^1) \eta(x') [-a^{ij}(t) u_{x^i x^j}(t, x)] \nu(dx) \right|^p dt \\
&= \nu(\mathcal{B}_r(a)) r^{2p} \int_0^{r^2} \left| \int_{\mathcal{B}_r(a)} \zeta(x^1) \eta_{x^j}(x') a^{ij}(t) u_{x^i}(t, x) \nu(dx) \right|^p dt \\
&\leq \nu(\mathcal{B}_r(a))^p r^{2p} \int_0^{r^2} \int_{\mathcal{B}_r(a)} \left| \zeta(x^1) \eta_{x^j}(x') a^{ij}(t) u_{x^i}(t, x) \right|^p \nu(dx) dt \\
&\leq N \nu(\mathcal{B}_r(a))^p r^{2p} \cdot \sup |\zeta|^p \cdot r^{-dp} \int_0^{r^2} \int_{\mathcal{B}_r(a)} |u_x|^p \nu(dx) dt \\
&\leq Nr^p \int_{\mathcal{Q}_r(a)} |u_x|^p \mu(dxdx).
\end{aligned}$$

For the integral with $f_{x^i}^i$ we use similar calculation to the one used to handle the term $-a^{ij} u_{x^i x^j}$, and get for each i

$$\begin{aligned}
& \nu(\mathcal{B}_r(a)) r^{2p} \int_0^{r^2} \left| \int_{\mathcal{B}_r(a)} \zeta(x^1) \eta(x') f_{x^i}^i(t, x) \nu(dx) \right|^p dt \\
&\leq N r^p \int_{\mathcal{Q}_r(a)} |f(t, x)|^p \mu(dtdx).
\end{aligned}$$

Hence, the lemma is proved. \square

Lemma 4.4. ([12]) Let $p \in [1, \infty)$, $0 < r \leq a$ and $u \in C_{loc}^\infty(\Omega)$.

(i) There is a constant $N = N(\theta, p, d, \delta, K)$ such that for any $\ell = 1, \dots, d$ we have

$$\int_{\mathcal{Q}_r(a)} |u_{x^\ell}(t, x) - (u_{x^\ell})_{\mathcal{Q}_r(a)}|^p \mu(dtdx) \leq Nr^p \int_{\mathcal{Q}_r(a)} (|u_{xx}(t, x)|^p + |u_t(t, x)|^p) \mu(dtdx). \quad (4.9)$$

(ii) Denote $\kappa_0 = \kappa_0(r, a) := (\nu^1((a-r, a+r))^{-1} \cdot \int_{a-r}^{a+r} x^1 \nu^1(dx^1))$. Then

$$\begin{aligned}
& \int_{\mathcal{Q}_r(a)} \left| u(t, x) - u_{\mathcal{Q}_r(a)} + \kappa_0(u_{x^1})_{\mathcal{Q}_r(a)} - \sum_{i=1}^d x^i (u_{x^i})_{\mathcal{Q}_r(a)} \right|^p \mu(dtdx) \\
&\leq Nr^p \int_{\mathcal{Q}_r(a)} (|u_x(t, x) - (u_x)_{\mathcal{Q}_r(a)}|^p + r^p |u_t(t, x)|^p + r^p |u_{xx}(t, x)|^p) \mu(dtdx) \\
&\leq Nr^{2p} \int_{\mathcal{Q}_r(a)} (|u_{xx}(t, x)|^p + |u_t(t, x)|^p) \mu(dtdx). \quad (4.10)
\end{aligned}$$

Proof. (i) For (4.9) we use that fact that $v := u_{x^\ell}$ satisfies $v_t - a^{ij} v_{x^i x^j} = (f^i)_{x^i}$, where $f^i = \delta^{i\ell} (u_t - a^{jm} u_{x^j x^m})$, and apply Lemma 4.3.

(ii) To prove (4.10), denote $v(t, x) := u(t, x) - (u)_{\mathcal{Q}_r(a)} + \kappa_0(u_{x^1})_{\mathcal{Q}_r(a)} - \sum_i x^i (u_{x^i})_{\mathcal{Q}_r(a)}$. Then

$$v_{\mathcal{Q}_r(a)} = \kappa_0(u_{x^1})_{\mathcal{Q}_r(a)} - \sum_i \frac{(u_{x^i})_{\mathcal{Q}_r(a)}}{\mu(\mathcal{Q}_r(a))} \int_{\mathcal{Q}_r(a)} x^i \nu(dx) dt = 0,$$

$$v - v_{\mathcal{Q}_r(a)} = v, \quad v_{x^i} = u_{x^i} - (u_{x^i})_{\mathcal{Q}_r(a)}, \quad v_t - a^{ij} v_{x^i x^j} = g := u_t - a^{ij} u_{x^i x^j}.$$

Now it is enough to use Lemma 4.3 and (4.9). The lemma is proved. \square

Theorem 4.5. Let $\theta \in (d-1, d-1+p)$, $0 < r \leq a$ and $\nu r/a \geq 2$. Assume that $u \in C_{loc}^\infty(\Omega)$ satisfies $u_t + a^{ij}(t)u_{x^i x^j} = 0$ in $\mathcal{Q}_{\nu r}(t_0, a, x'_0) \cap \Omega$. Then there is a constant $N = N(K, \delta, \theta, p, d)$ so that

$$\begin{aligned} & \int_{\mathcal{Q}_r(t_0, a, x'_0)} |u_{xx}(t, x) - (u_{xx})_{\mathcal{Q}_r(t_0, a, x'_0)}|^p \mu(dt dx) \\ & \leq \frac{N}{(1 + \nu r/a)^p} \int_{\mathcal{Q}_{\nu r}(t_0, a, x'_0) \cap \Omega} |u_{xx}(t, x)|^p \mu(dt dx). \end{aligned} \quad (4.11)$$

Proof. Considering a proper translation, without loss of generality, we assume that $t_0 = 0$, $x'_0 = 0$ and thus $\mathcal{Q}_r(t_0, a, x'_0) = \mathcal{Q}_r(a)$.

Step 1. First, we consider the case $a = 1$. Obviously,

$$r \leq 1, \quad 2 \leq \nu r, \quad \beta := \frac{1 + \nu r}{2} \leq \nu r, \quad \frac{r}{\beta} \leq \frac{1}{\beta} \leq \frac{2}{3}, \quad 2\beta = 1 + \nu r.$$

Thus,

$$\mathcal{Q}_\beta(\beta) \subset \mathcal{Q}_{\nu r}(1) \cap \Omega, \quad \mathcal{Q}_{r/\beta}(\beta^{-1}) \subset \mathcal{Q}_{2/3}(2/3).$$

Denote $w(t, x) = u(\beta^2 t, \beta x)$, then obviously

$$w_t + a^{ij}(\beta^2 t)w_{x^i x^j} = 0, \quad \text{for } (t, x) \in \mathcal{Q}_1(1)$$

and

$$\begin{aligned} \int_{\mathcal{Q}_r(1)} |u_{xx}(t, x) - (u_{xx})_{\mathcal{Q}_r(1)}|^p \mu(dt dx) & \leq N(d) \sup_{\mathcal{Q}_r(1)} (|u_{xxx}|^p + |u_{xxt}|^p) \\ & \leq N(d) \beta^{-3p} \sup_{\mathcal{Q}_{r/\beta}(\beta^{-1})} (|w_{xxx}|^p + |w_{xxt}|^p) \\ & \leq N(d) \beta^{-3p} \sup_{\mathcal{Q}_{2/3}(2/3)} (|w_{xxx}|^p + |w_{xxt}|^p). \end{aligned}$$

Applying Lemma 3.7 to $v(t, x) = w(t, x) - w_{\mathcal{Q}_1(1)} + \kappa_0(w_{x^1})_{\mathcal{Q}_1(1)} - \sum_{i=1}^d x^i (w_{x^i})_{\mathcal{Q}_1(1)}$, and then using Lemma 4.4

$$\begin{aligned} \beta^{-3p} \sup_{\mathcal{Q}_{2/3}(2/3)} (|w_{xxx}|^p + |w_{xxt}|^p) & \leq N \beta^{-3p} \int_{\mathcal{Q}_1(1)} |v|^p \mu(dt dx) \\ & \leq N \beta^{-3p} \int_{\mathcal{Q}_1(1)} |w_{xx}|^p \mu(dt dx) \\ & = N \beta^{-2p-2-\theta} \int_{\mathcal{Q}_\beta(\beta)} |u_{xx}|^p \mu(dt dx). \end{aligned}$$

This leads to (4.11) since $|\mathcal{Q}_{\nu r}(1) \cap \Omega| \sim \beta^{p+\theta+2}$.

Step 2. Let $a \neq 1$. Define $v(t, x) := u(a^2 t, ax)$. Then $v_t + a^{ij}(a^2 t)v_{x^i x^j} = 0$ in $\mathcal{Q}_{\nu r/a}(1) \cap \Omega$. It is easy to check

$$\mu(\mathcal{Q}_{r/a}(1)) = a^{-\theta-p-2} \mu(\mathcal{Q}_r(a)), \quad (v_{xx})_{\mathcal{Q}_{r/a}(1)} = a^2 (u_{xx})_{\mathcal{Q}_r(a)}, \quad \mu(\mathcal{Q}_{\nu r/a}(1) \cap \Omega) = a^{-\theta-p-2} \mu(\mathcal{Q}_{\nu r}(a) \cap \Omega),$$

and consequently

$$\int_{\mathcal{Q}_{r/a}(1)} |v_{xx}(t, x) - (v_{xx})_{\mathcal{Q}_{r/a}(1)}|^p \mu(dt dx) = a^{2p} \int_{\mathcal{Q}_r(a)} |u_{xx}(t, x) - (u_{xx})_{\mathcal{Q}_r(a)}|^p \mu(dt dx),$$

$$\int_{\mathcal{Q}_{\nu r/a}(1) \cap \Omega} |v_{xx}(t, x)|^p \mu(dt dx) = a^{2p} \int_{\mathcal{Q}_{\nu r}(a) \cap \Omega} |u_{xx}(t, x)|^p \mu(dt dx).$$

It follows that

$$\begin{aligned} \int_{\mathcal{Q}_r(a)} |u_{xx}(t, x) - (u_{xx})_{\mathcal{Q}_r(a)}|^p \mu(dt dx) &= a^{-2p} \int_{\mathcal{Q}_{\nu r/a}(1) \cap \Omega} |v_{xx}(t, x)|^p \mu(dt dx) \\ &\leq a^{-2p} \frac{N}{(1 + \nu r/a)^p} \int_{\mathcal{Q}_{\nu r/a}(1) \cap \Omega} |v_{xx}(t, x)|^p \mu(dt dx) \\ &= \frac{N}{(1 + \nu r/a)^p} \int_{\mathcal{Q}_{\nu r}(a) \cap \Omega} |u_{xx}(t, x)|^p \mu(dt dx). \end{aligned}$$

The theorem is proved. \square

The following is the main result of this section. Recall that $\theta < d - 1 + p$. Thus for q sufficiently close to p , we have $\theta + p - q < d - 1 + q$.

Theorem 4.6. *Let $\theta \in (d - 1, d - 1 + p)$, $0 < r \leq a$ and $p, q \in (1, \infty)$ so that*

$$q \leq p, \quad \theta' := \theta + p - q < d - 1 + q. \quad (4.12)$$

Also let $\nu \geq 2$, $\nu r \geq a$ and $u \in C^\infty(\Omega)$. Then,

$$\begin{aligned} &\int_{\mathcal{Q}_r(t_0, a, x_0)} |u_{xx}(t, x) - (u_{xx})_{\mathcal{Q}_r(t_0, a, x_0)}|^q \mu(dt dx) \\ &\leq N \frac{1}{(1 + \nu r/a)^q} \int_{\mathcal{Q}_{\nu r}(t_0, a, x'_0) \cap \Omega} |u_{xx}(t, x)|^q \mu(dt dx) \\ &+ N \frac{\nu^{d+1}}{r/a} (1 + \nu r/a)^{p+\theta-d+1} \int_{\mathcal{Q}_{\nu r}(t_0, a, x'_0) \cap \Omega} |u_t + a^{ij} u_{x^i x^j}|^q \mu(dt dx), \end{aligned}$$

where $N = N(K, \delta, \theta, p, q)$.

Proof. As before we may assume that $t_0 = 0$ and $x'_0 = 0$. Also we may assume that $a^{ij}(t)$ is infinitely differentiable in t and all the derivatives of a^{ij} are bounded. Indeed, take a sequence of smooth functions a_n^{ij} so that each a_n^{ij} satisfies condition (3.2) and $a_n^{ij}(t) \rightarrow a^{ij}(t)$ as $n \rightarrow \infty$ (a.e.). Then it is enough to observe

$$\int_{\mathcal{Q}_{\nu r}(t_0, a, x'_0) \cap \Omega} |u_t + a_n u_{xx}|^q \mu(dt dx) \rightarrow \int_{\mathcal{Q}_{\nu r}(t_0, a, x'_0) \cap \Omega} |u_t + a u_{xx}|^q \mu(dt dx) \quad \text{as } n \rightarrow \infty.$$

Also note that we may assume that $u(t)$ vanishes for all large t , say for all $t \geq T (\geq \nu^2 r^2)$.

Take a $\zeta \in C_0^\infty(\mathbb{R}^{d+1})$ so that $\zeta(t, x) = 1$ for $(t, x) \in \mathcal{Q}_{\nu r/2}(a) \cap \Omega$ and $\zeta(t, x) = 0$ if $(t, x) \notin (-\nu^2 r^2, \nu^2 r^2) \times (-a, a + \nu r) \times B'_{\nu r}$. Denote

$$f = u_t + a^{ij} u_{x^i x^j}, \quad g = f\zeta, \quad h = f(1 - \zeta).$$

By Corollary 3.4 we can define v as the solution of

$$v_t + a^{ij} v_{x^i x^j} = h \quad \text{for } t \in (-\infty, T), \quad \text{and } v(T, \cdot) = 0 \quad (4.13)$$

so that $v \in M\mathbb{H}_{p,\theta}^n(-\infty, T)$ for any n . Also let $\bar{v} \in M\mathbb{H}_{p,\theta}^n(-\infty, T+1)$ be the solution of

$$\bar{v}_t + a^{ij}\bar{v}_{x^i x^j} = h \quad \text{for } t \in (-\infty, T+1), \quad \text{and } \bar{v}(T+1, \cdot) = 0.$$

Then by considering the equation for \bar{v} on $(T, T+1)$, since $h(t) = 0$ for $t \geq T$, we conclude $\bar{v}(t) = 0$ for $t \in [T, T+1]$. Thus \bar{v} also satisfies (4.13) and $v = \bar{v}$. It follows from (2.9) that v is infinitely differentiable in x (and hence in t), and thus $v \in C_{loc}^\infty(\Omega)$.

By (4.12),

$$\theta - d + p = \theta' - d + q, \quad \theta' \in (d-1, d-1+q).$$

By applying Theorem 4.5 with q, θ' and $\nu/2$ in places of p, θ and ν respectively,

$$\begin{aligned} \int_{\mathcal{Q}_r(a)} |v_{xx}(t, x) - (v_{xx})_{\mathcal{Q}_r(a)}|^q \bar{\mu}(dyds) &\leq N \frac{1}{(1 + \nu r/2a)^q} \int_{\mathcal{Q}_{\nu r/2}(a) \cap \Omega} |v_{xx}(t, x)|^q \bar{\mu}(dyds) \\ &\leq N \frac{1}{(1 + \nu r/a)^q} \int_{\mathcal{Q}_{\nu r}(a) \cap \Omega} |v_{xx}(t, x)|^q \bar{\mu}(dyds), \end{aligned} \quad (4.14)$$

where $\bar{\mu}(dsdy) := (y^1)^{\theta' - d + q} dyds = \mu(dyds)$. On the other hand, $w := u - v$ satisfies

$$w_t + a^{ij}w_{x^i x^j} = g, \quad t \in (0, T).$$

and $w(T, \cdot) = 0$. By Corollary 3.4 (with q, θ' in place of p, θ respectively),

$$\begin{aligned} \int_{\mathcal{Q}_r(a)} |w_{yy}|^q (y^1)^{\theta' - d + q} dyds &\leq \int_{\mathcal{Q}_{\nu r}(a) \cap \Omega} |w_{yy}|^q \mu(dsdy) \leq N \int_{\mathcal{Q}_{\nu r}(a) \cap \Omega} |f|^q \mu(dsdy), \\ \int_{\mathcal{Q}_r(a)} |w_{yy}|^q \mu(dyds) &\leq N \frac{\nu^{d+1} (1 + \nu r/a)^{p+\theta-d+1}}{(1 + r/a)^{p+\theta-d+1} - (1 - r/a)^{p+\theta-d+1}} \int_{\mathcal{Q}_{\nu r}(a) \cap \Omega} |f|^q \mu(dyds) \\ &\leq N \frac{a}{r} \nu^{d+1} (1 + \nu r/a)^{p+\theta-d+1} \int_{\mathcal{Q}_{\nu r}(a) \cap \Omega} |f|^q \mu(dyds), \end{aligned} \quad (4.15)$$

where inequality (4.15) is obtained as follows; since $p + \theta - d + 1 \geq 1$,

$$(1 + r/a)^{p+\theta-d+1} - (1 - r/a)^{p+\theta-d+1} \geq (1 + r/a) - (1 - r/a) \geq 2r/a.$$

Observing that $u = v + w$, we get

$$\begin{aligned} I &:= \int_{\mathcal{Q}_r(a)} |u_{yy}(t, x) - (u_{yy})_{\mathcal{Q}_r(a)}|^q \mu(dyds) \\ &\leq N(q) \int_{\mathcal{Q}_r(a)} |w_{yy}(t, x) - (w_{yy})_{\mathcal{Q}_r(a)}|^q \mu(dyds) + N(q) \int_{\mathcal{Q}_r(a)} |v_{yy}(t, x) - (v_{yy})_{\mathcal{Q}_r(a)}|^q \mu(dyds) \\ &\leq N(q) \int_{\mathcal{Q}_r(a)} |w_{yy}(t, x)|^q \mu(dyds) + N(q) \int_{\mathcal{Q}_r(a)} |v_{yy}(t, x) - (v_{yy})_{\mathcal{Q}_r(a)}|^q \mu(dyds). \end{aligned}$$

Thus by (4.14) and (4.15),

$$\begin{aligned}
I &\leq N \frac{a}{r} \nu^{d+1} (1 + \nu r/a)^{p+\theta-d+1} \int_{\mathcal{Q}_{\nu r(a)} \cap \Omega} |f|^q \mu(dyds) + N \frac{1}{(1 + \nu r/a)^q} \int_{\mathcal{Q}_{\nu r(a)} \cap \Omega} |v_{yy}(t, x)|^q \mu(dyds) \\
&\leq N \frac{a}{r} \nu^{d+1} (1 + \nu r/a)^{p+\theta} \int_{(0, \nu^2 r^2) \times (0, a + \nu r)} |f|^q \mu(dyds) \\
&\quad + N \frac{1}{(1 + \nu r/a)^q} \int_{\mathcal{Q}_{\nu r(a)} \cap \Omega} (|u_{yy}(t, x)|^q + |w_{yy}(t, x)|^q) \mu(dyds) \\
&\leq N \frac{a}{r} \nu^{d+1} (1 + \nu r/a)^{p+\theta-d+1} \int_{\mathcal{Q}_{\nu r(a)} \cap \Omega} |f|^q \mu(dyds) + N \frac{1}{(1 + \nu r/a)^q} \int_{\mathcal{Q}_{\nu r(a)} \cap \Omega} |u_{yy}(t, x)|^q \mu(dyds).
\end{aligned}$$

The theorem is proved. \square

5 A priori estimate for equations with BMO coefficients

Recall that $\mathcal{Q}_r(t, x) = (t, t + r^2) \times (x^1 - r, x^1 + r) \times B'_r(x')$. For any $d \times d$ matrix $a = (a^{ij}(t, x))$, as in [17], we define a standard mean oscillation on $\mathcal{Q}_r(t, x) = \mathcal{Q}_r(t, x^1, x')$:

$$osc_x(a, \mathcal{Q}_r(t, x)) = \frac{1}{r^2 |\mathcal{B}_r(x)|^2} \int_t^{t+r^2} \left(\int_{\mathcal{B}_r(x)} \int_{\mathcal{B}_r(x)} |a(s, y) - a(s, z)| dydz \right) ds, \quad (5.1)$$

where $|\mathcal{B}_r(x)|$ is the Euclidian volume of $\mathcal{B}_r(x)$. We say that a is VMO (see [17] for more details) if

$$\lim_{r \rightarrow 0} \sup_{\mathcal{Q}_r(t, x)} osc_x(a, \mathcal{Q}_r(t, x)) = 0. \quad (5.2)$$

Now we define a mean oscillation with respect to measure $\nu(dx) = (x^1)^{\theta-d+p} dx$:

$$\begin{aligned}
osc_x^\theta(a, \mathcal{Q}_r(t, x^1, x')) &= r^{-2} \int_t^{t+r^2} \left(\int_{\mathcal{B}_r(x)} \int_{\mathcal{B}_r(x)} |a(s, y) - a(s, z)| \nu(dy) \nu(dz) \right) ds \\
&= \frac{1}{r^2 (\nu(\mathcal{B}_r(x)))^2} \int_t^{t+r^2} \left(\int_{\mathcal{B}_r(x)} \int_{\mathcal{B}_r(x)} |a(s, y) - a(s, z)| \nu(dy) \nu(dz) \right) ds.
\end{aligned}$$

Obviously, if a depends only on t then $osc_x^\theta(a, \mathcal{Q}_r(t, x^1, x')) = 0$. Also it is easy to check that for any $d \times d$ matrix-valued $\bar{a}(t)$ depending only on t ,

$$osc_x^\theta(a, \mathcal{Q}_r(t, x)) \leq 2r^{-2} \int_t^{t+r^2} \int_{\mathcal{B}_r(x)} |a(s, y) - \bar{a}(s)| \nu(dy) ds.$$

On the other hand,

$$r^{-2} \int_t^{t+r^2} \int_{\mathcal{B}_r(x)} |a(s, y) - (a)_{\mathcal{B}_r(x)}(s)| \nu(dy) ds \leq osc_x^\theta(a, \mathcal{Q}_r(t, x)),$$

where $(a)_{\mathcal{B}_r(x)}(s) = (\nu(\mathcal{B}_r(x)))^{-1} \int_{\mathcal{B}_r(x)} a(s, y) \nu(dy)$.

Roughly speaking, the following result says that the condition $osc_x^\theta(a, \mathcal{Q}_r(t, x^1, x')) \leq \varepsilon$ for some ε is not stronger than the condition $osc_x(a, \mathcal{Q}_r(t, x^1, x')) \leq \varepsilon$.

Lemma 5.1. *There exists a constant $N = N(\theta) > 0$ so that for any $\kappa \in (0, 1]$ and $r = \kappa x^1$,*

$$\text{osc}_x^\theta(a, \mathcal{Q}_r(t, x^1, x')) \leq N \text{osc}_x(a, \mathcal{Q}_r(t, x^1, x')), \quad (5.3)$$

$$\text{osc}_x(a, \mathcal{Q}_r(t, x^1, x')) \leq N \cdot (1 - \kappa)^{-\alpha} \text{osc}_x^\theta(a, \mathcal{Q}_r(t, x^1, x')). \quad (5.4)$$

Proof. Denote $\alpha := \theta - d + p > -1$. First note that

$$\begin{aligned} \nu(dy) &\leq (x^1)^\alpha (1 + \kappa)^\alpha dy \quad \text{on } \mathcal{B} := \mathcal{B}_r(x), \\ \frac{|\mathcal{B}|}{\nu(\mathcal{B})} &= (\alpha + 1)(x^1)^{-\alpha} \frac{2\kappa}{[(1 + \kappa)^{\alpha+1} - (1 - \kappa)^{\alpha+1}]} \leq N(\alpha)(x^1)^{-\alpha}, \end{aligned}$$

where the last inequality is obtained as in (4.15). Thus

$$\begin{aligned} \text{osc}_x^\theta(a, \mathcal{Q}_r(t, x^1, x')) &= \frac{|\mathcal{B}|^2}{(\nu(\mathcal{B}))^2} \frac{r^{-2}}{|\mathcal{B}|^2} \int_t^{t+r^2} \left(\int_{\mathcal{B}} \int_{\mathcal{B}} |a(s, y) - a(s, z)| \nu(dy) \nu(dz) \right) ds \\ &\leq N^2(\alpha)(x^1)^{-2\alpha} \cdot (x^1)^{2\alpha} (1 + \kappa)^{2\alpha} \text{osc}_x(a, \mathcal{Q}_r(t, x^1, x')) \\ &\leq N \text{osc}_x(a, \mathcal{Q}_r(t, x^1, x')). \end{aligned}$$

To prove (5.4) it is enough to assume $\kappa \in (0, 1)$ and note $dy \leq \frac{\nu(dy)}{(x^1)^\alpha (1 - \kappa)^\alpha}$ on $\mathcal{B}_r(x)$. The lemma is proved. \square

Remark 5.2. In Theorem 6.6, the following condition near $\partial\mathbb{R}_+^d$ is assumed:

$$\lim_{x^1 \rightarrow 0} \sup_{r \leq \kappa_0 x^1} \text{osc}_x^\theta(a, \mathcal{Q}_r(t, x^1, x')) < \varepsilon, \quad (5.5)$$

where $\kappa_0, \varepsilon \in (0, 1)$ will be specified later. To understand (5.5), let $d = 1$ (so that $\mathbb{R}_+^d = (0, \infty)$) and a be independent of t . Then (5.5) becomes

$$\lim_{x^1 \rightarrow 0} \sup_{r \leq \kappa_0 x^1} \text{osc}_x^\theta(a, (x^1 - r, x^1 + r)) < \varepsilon. \quad (5.6)$$

Since $x^1 - r \geq (1 - \kappa_0)x^1 > 0$, there is no requirement that the mean oscillation on a ball containing the boundary points is small. **Obviously (5.5) is satisfied if a is VMO**, since (cf. (5.3))

$$\lim_{x^1 \rightarrow 0} \sup_{r \leq \kappa_0 x^1} \text{osc}_x^\theta(a, \mathcal{Q}_r(t, x^1, x')) \leq N \lim_{x^1 \rightarrow 0} \sup_{r \leq x^1} \text{osc}_x(a, \mathcal{Q}_r(t, x)) = 0.$$

Note that in the following result $dxdt$ is used in place of $\mu(dxdt)$. However, if r/a is small then $\int_{\mathcal{Q}_r(a)} dt dx$ and $\int_{\mathcal{Q}_r(a)} \mu(dt dx)$ are comparable.

Lemma 5.3. *Let $q > 1$ and $a^{ij} = a^{ij}(t)$. Then there exists a constant $N = N(\delta, K, p, d)$ so that for any $\nu \geq 4$, $r > 0$ and $u \in C^\infty(\Omega)$,*

$$\int_{\mathcal{Q}_r(a)} \int_{\mathcal{Q}_r(a)} |u_{xx}(t, x) - u_{xx}(s, y)|^q dx dt dy ds \leq N \nu^{-q} \int_{\mathcal{Q}_{\nu r}(a)} |u_{xx}|^q dx dt + N \nu^{d+2} \int_{\mathcal{Q}_{\nu r}(a)} |u_t + a^{ij} u_{x^i x^j}|^q dx dt$$

Proof. See Theorem 6.1.2 of [16]. \square

For $\kappa \in (0, 1]$ and $R > 0$, let $\mathbb{Q}(R, \kappa)$ be the collection of all $\mathcal{Q}_r(t, x)$ so that $r \leq \kappa x^1$ and $\mathcal{Q}_r(t, x) \subset \{(t, y) \in \Omega : y^1 \in (0, R)\}$. That is, $\mathcal{Q}_r(t, x) \in \mathbb{Q}(R, \kappa)$ if

$$x^1 > 0, \quad r \leq \kappa x^1, \quad x^1 + r \leq R.$$

Define

$$a_{R, \kappa}^{\#(\theta)} = \sup_{\mathcal{Q} \in \mathbb{Q}(R, \kappa)} \text{osc}_x^\theta(a, \mathcal{Q}), \quad a_\kappa^{\#(\theta)} = \sup_{R > 0} a_{R, \kappa}^{\#(\theta)}.$$

Lemma 5.4. *Let $\beta \in (1, \infty)$, $\kappa \in (1/2, 1)$ and*

$$1 < q < p, \quad \theta + p - q < d - 1 + q.$$

Suppose that $u \in C^\infty(\Omega)$ vanishes outside $\mathcal{Q}_0 \in \mathbb{Q}(R, \kappa)$. Then for any $\varepsilon > 0$, $\mathcal{Q}_r(t_1, a, x'_1) \subset \Omega$ and $(t, x) \in \mathcal{Q}_r(t_1, a, x'_1)$ we have

$$\begin{aligned} & \int_{\mathcal{Q}_r(t_1, a, x'_1)} |u_{xx} - (u_{xx})_{\mathcal{Q}_r(t_1, a, x'_1)}|^q \mu(dyds) \\ & \leq \varepsilon \mathbb{M}(|u_{xx}|^q)(t, x) + N \mathbb{M}(|f|^q)(t, x) + N (a_{2R, \kappa}^{\#(\theta)})^{1/\beta'} \cdot \mathbb{M}^{1/\beta}(|u_{xx}|^{\beta q})(t, x) \end{aligned} \quad (5.7)$$

where $f := u_t + a^{ij} u_{x^i x^j}$, $\beta' := \beta/(\beta - 1)$ and $N = N(\varepsilon, \theta, q, d, \delta, K)$.

Proof. Let $\mathcal{Q}_0 = \mathcal{Q}_{r_0}(t_0, a_0, x'_0)$. Considering a translation, we may assume $t_1 = 0$, $x'_1 = 0 \in \mathbb{R}^{d-1}$ so that $\mathcal{Q}_r(t_1, a, x'_1) = \mathcal{Q}_r(a)$. Also, we assume

$$\mathcal{Q}_r(a) \cap \mathcal{Q}_0 \neq \emptyset. \quad (5.8)$$

Otherwise, the left term of (5.7) becomes zero.

Step 1. Firstly, we prove that there exists $\delta_0 = \delta_0(\varepsilon) \in (0, 1)$ so that (5.7) holds if $r/a \leq \delta_0$. Let $|\mathcal{Q}|$ denote the Lebesgue measure of $\mathcal{Q} \subset \mathbb{R}^{d+1}$. Assume $\nu \geq 4$ and $\nu r \leq a/4$. Then $(3a/4) \leq x^1 \leq (5a/4)$ if $x^1 \in \mathcal{B}_{\nu r}^1(a) := (a - \nu r, a + \nu r)$. Denote $c_0 := \left(\frac{5}{3}\right)^{\theta-d+p}$, then

$$\frac{\mu(dtdx)}{\mu(\mathcal{Q}_r(a))} \leq c_0 \frac{dtdx}{|\mathcal{Q}_r(a)|} \quad \text{on } \mathcal{Q}_r(a),$$

$$\frac{dtdx}{|\mathcal{Q}_{\nu r}(a)|} \leq c_0 \frac{\mu(dtdx)}{\mu(\mathcal{Q}_{\nu r}(a))} \quad \text{on } \mathcal{Q}_{\nu r}(a).$$

Also due to (5.8), we have $a - r < a_0 + r_0$ and thus

$$a + \nu r \leq 2R, \quad \frac{\nu r}{a} \leq 1/4 \leq \kappa, \quad \mathcal{Q}_{\nu r}(a) \subset \mathbb{Q}_{2R, \kappa}. \quad (5.9)$$

Denote $\bar{a}^{ij}(t) = (a^{ij}(t, \cdot))_{B_{\nu r}(a)}$ and $f = u_t + a^{ij}u_{x^i x^j}$. By Lemma 5.3,

$$\begin{aligned}
& \int_{\mathcal{Q}_r(a)} |u_{xx} - (u_{xx})_{\mathcal{Q}_r(a)}|^q \mu(dsdy) \\
& \leq \frac{1}{(\mu(\mathcal{Q}_r(a)))^2} \int_{\mathcal{Q}_r(a)} \int_{\mathcal{Q}_r(a)} |u_{xx}(s, y) - u_{xx}(\tau, \xi)|^q \mu(dsdy) \mu(d\tau d\xi) \\
& \leq c_0^2 \frac{1}{|\mathcal{Q}_r(a)|^2} \int_{\mathcal{Q}_r(a)} \int_{\mathcal{Q}_r(a)} |u_{xx}(s, y) - u_{xx}(\tau, \xi)|^q dsdy d\tau d\xi \tag{5.10} \\
& \leq Nc_0^2 \nu^{d+2} \int_{\mathcal{Q}_{\nu r}(a)} |u_t + \bar{a}^{ij}u_{x^i x^j}|^q \frac{dyds}{|\mathcal{Q}_{\nu r}(a)|} + Nc_0^2 \nu^{-q} \int_{\mathcal{Q}_{\nu r}(a)} |u_{xx}|^q \frac{dyds}{|\mathcal{Q}_{\nu r}(a)|} \\
& \leq Nc_0^3 \nu^{d+2} \int_{\mathcal{Q}_{\nu r}(a)} |f|^q \mu(dyds) + Nc_0^3 \nu^{d+2} \cdot J + Nc_0^3 \nu^{-q} \int_{\mathcal{Q}_{\nu r}(a)} |u_{xx}|^q \mu(dyds), \tag{5.11}
\end{aligned}$$

where $N = N(d, \delta, K)$ and

$$J := \int_{\mathcal{Q}_{\nu r}(a)} |(a^{ij} - \bar{a}^{ij})u_{x^i x^j}|^q \mu(dt dx) \leq NJ_1^{1/\beta} J_2^{1/\beta'},$$

$$J_1 := \int_{\mathcal{Q}_{\nu r}(a)} |u_{xx}|^{q\beta} \mu(dt dx) \leq N\mathbb{M}(|u_{xx}|^{\beta q})(t, x),$$

$$J_2 := \int_{\mathcal{Q}_{\nu r}(a)} |a^{ij} - \bar{a}^{ij}|^{q\beta'} \mu(dt dx) \leq N \int_{\mathcal{Q}_{\nu r}(a)} |a^{ij} - \bar{a}^{ij}| \mu(dt dx) \tag{5.12}$$

$$\leq Na_{2R, \kappa}^{\#(\theta)}, \tag{5.13}$$

where inequality (5.12) is due to $|a^{ij}| \leq K$, and (5.9) is used in (5.13). Coming back to (5.11), we get

$$\begin{aligned}
& \int_{\mathcal{Q}_r(a)} |u_{xx} - (u_{xx})_{\mathcal{Q}_r(a)}|^q \mu(dsdy) \\
& \leq N\nu^{d+2} \mathbb{M}(|f|^q)(t, x) + N\nu^{-q} \mathbb{M}(|u_{xx}|^q)(t, x) + N\nu^{d+2} (a_{2R, \kappa}^{\#(\theta)})^{1/\beta'} \mathbb{M}^{1/\beta}(|u_{xx}|^{q\beta})(t, x).
\end{aligned}$$

Remember that the above inequality holds whenever $\nu \geq 4$ and $r/a \leq (4\nu)^{-1}$. Now we fix ν so that $N\nu^{-q} \leq \varepsilon$ and take $\delta_0 = 1/(4\nu)$. Then whenever $r/a \leq \delta_0$ we have $(r/a)\nu \leq 1/4$ and thus (5.7) follows.

Step 2. For given ε , take $\delta_0 = \delta_0(\varepsilon)$ from Step 1. Assume $r/a \geq \delta_0$. Choose ν , which will be specified later, so that $r\nu > 4a$. Denote $\alpha := \theta - d + p$.

Here we claim that if $\mu(\mathcal{Q}_0) \geq 2^{d+2\alpha+3} \mu(\mathcal{Q}_{\nu r}(a) \cap \Omega)$, then for $\bar{a} := x_0 - r_0 + \nu r$ we have

$$(\mathcal{Q}_{r_0}(t_0, x_0, x'_0) \cap \mathcal{Q}_{\nu r}(a)) \subset \mathcal{Q}_{\nu r}(\bar{a}), \quad \bar{a} + \nu r \leq 2R, \quad \nu r/\bar{a} \leq \kappa, \quad |\mathcal{Q}_{\nu r}(\bar{a})| \leq 2^{\alpha+1} |\mathcal{Q}_{\nu r}(a) \cap \Omega|. \tag{5.14}$$

First, due to (5.8), we have $0 < x_0 - r_0 < 2a$. Let ω_{d-1} denote the volume of $B'_1(0)$. Then

$$\mu(\mathcal{Q}_0) \leq |\mathcal{Q}_{x_0}(t_0, x_0, x'_0)| = \frac{1}{\alpha+1} \omega_{d-1} 2^{\alpha+1} x_0^{\alpha+d+2},$$

$$|\mathcal{Q}_{\nu r}(a) \cap \Omega| = \frac{1}{\alpha+1} \omega_{d-1} (a + \nu r)^{\alpha+1} (\nu r)^{d+1} \geq \frac{1}{\alpha+1} \omega_{d-1} (\nu r)^{\alpha+d+2}.$$

Thus, by assumption it follows that $2^{\alpha+1}x_0^{d+\alpha+2} \geq 2^{d+2\alpha+3}(\nu r)^{\alpha+d+2}$ or equivalently $x_0 \geq 2\nu r$. Observe

$$(\mathcal{Q}_{r_0}(t_0, x_0, x'_0) \cap \mathcal{Q}_{\nu r}(a)) \subset ((0, (\nu r)^2) \times (x_0 - r_0, a + \nu r) \times B'_{\nu r}) \subset \mathcal{Q}_{\nu r}(\bar{a}).$$

Also from the inequality $r_0 \geq 1/2x_0 \geq \nu r$ (recall $\kappa \geq 1/2$), we get

$$\frac{\nu r}{\bar{a}} = \frac{\nu r}{x_0 - r_0 + \nu r} \leq \frac{r_0}{x_0} \leq \kappa.$$

Since the last inequality of (5.14) is obvious, the claim is proved. Note that (5.14) implies that $\mathcal{Q}_{\nu r}(\bar{a}) \in \mathbb{Q}_{2R, \kappa}$.

Now define $\bar{a}^{ij} = (a^{ij})_{\mathcal{Q}_{r_0}(t_0, x_0, x'_0)}$ if $|\mathcal{Q}_{r_0}(t_0, x_0, x'_0)| < 2^{d+2\alpha+3}|\mathcal{Q}_{\nu r}(a) \cap \Omega|$, and otherwise define $\bar{a}^{ij} = (a^{ij})_{\mathcal{Q}_{\nu r}(\bar{a})}$, where $\bar{a} = x_0 - r_0 + \nu r$ as defined above. By Theorem 4.6

$$\begin{aligned} & \int_{\mathcal{Q}_{r_0}(a)} |u_{xx}(t, x) - (u_{xx})_{\mathcal{Q}_{r_0}(a)}|^q \mu(dtdx) \\ & \leq \frac{N}{(1 + \nu r/a)^q} \int_{\mathcal{Q}_{\nu r}(a) \cap \Omega} |u_{xx}(t, x)|^q \mu(dtdx) + \frac{N \cdot \nu^{d+1}}{r/a} (1 + \nu r/a)^{p+\theta-d+1} \int_{\mathcal{Q}_{\nu r}(a) \cap \Omega} |u_t + \bar{a}u_{xx}|^q \mu(dtdx) \\ & \leq \frac{N}{(1 + \nu r/a)^q} \int_{\mathcal{Q}_{\nu r}(a) \cap \Omega} |u_{xx}(t, x)|^q \mu(dtdx) + \frac{N \cdot \nu^{d+1}}{r/a} (1 + \nu r/a)^{p+\theta-d+1} \left(\int_{\mathcal{Q}_{\nu r}(a) \cap \Omega} |f|^q \mu(dtdx) + J \right), \end{aligned}$$

where

$$\begin{aligned} J &:= \int_{\mathcal{Q}_{\nu r}(a) \cap \Omega} |(a^{ij} - \bar{a}^{ij})u_{x^i x^j}|^q \mu(dtdx) \leq N(\nu r)^{-\alpha-d-2} \int_{\mathcal{Q}_{\nu r}(a) \cap \Omega} |(a^{ij} - \bar{a}^{ij})u_{x^i x^j}|^q \mu(dtdx) \\ &= N(\nu r)^{-\alpha-d-2} \int_{\mathcal{Q}_{\nu r}(a) \cap \mathcal{Q}_{r_0}(t_0, x_0, x'_0)} |(a^{ij} - \bar{a}^{ij})u_{x^i x^j}|^q \mu(dtdx) \leq N(\nu r)^{-\alpha-d-2} J_1^{1/\beta} J_2^{1/\beta'}, \\ J_1 &:= \int_{\mathcal{Q}_{\nu r}(a) \cap \Omega} |u_{xx}|^{q\beta} \mu(dtdx) \leq N(\nu r)^{\alpha+d+2} \int_{\mathcal{Q}_{\nu r}(a) \cap \Omega} |u_{xx}|^{q\beta} \mu(dtdx) \leq N(\nu r)^{\alpha+d+2} \mathbb{M}(|u_{xx}|^{\beta q})(t, x), \\ J_2 &:= \int_{\mathcal{Q}_{\nu r}(a) \cap \mathcal{Q}_{r_0}(t_0, x_0, x'_0)} |a^{ij} - \bar{a}^{ij}|^{q\beta'} \mu(dtdx) \leq N \int_{\mathcal{Q}_{\nu r}(a) \cap \mathcal{Q}_{r_0}(t_0, x_0, x'_0)} |a^{ij} - \bar{a}^{ij}| \mu(dtdx). \end{aligned}$$

If $|\mathcal{Q}_{r_0}(t_0, x_0, x'_0)| < 2^{d+2\alpha+3}|\mathcal{Q}_{\nu r}(a) \cap \Omega|$, then

$$\begin{aligned} \int_{\mathcal{Q}_{\nu r}(a) \cap \mathcal{Q}_{r_0}(t_0, x_0, x'_0)} |a^{ij} - \bar{a}^{ij}| \mu(dtdx) &\leq \int_{\mathcal{Q}_{r_0}(t_0, x_0, x'_0)} |a^{ij} - (a^{ij})_{\mathcal{Q}_{r_0}(t_0, x_0, x'_0)}| \mu(dtdx) \\ &= \mu(\mathcal{Q}_0) \int_{\mathcal{Q}_{r_0}(t_0, x_0, x'_0)} |a^{ij} - (a^{ij})_{\mathcal{Q}_{r_0}(t_0, x_0, x'_0)}| \mu(dtdx) \\ &\leq N(\nu r)^{\alpha+d+2} a_{R, \kappa}^{\#(\theta)}, \end{aligned}$$

and if $|\mathcal{Q}_{r_0}(t_0, x_0, x'_0)| \geq 2^{d+2\alpha+3}|\mathcal{Q}_{\nu r}(a) \cap \Omega|$, then

$$\begin{aligned} \int_{\mathcal{Q}_{\nu r}(a) \cap \mathcal{Q}_{r_0}(t_0, x_0, x'_0)} |a^{ij} - \bar{a}^{ij}| \mu(dtdx) &\leq \int_{\mathcal{Q}_{\nu r}(\bar{a})} |a^{ij} - (a^{ij})_{\mathcal{Q}_{\nu r}(\bar{a})}| \mu(dtdx) \\ &= \mu(\mathcal{Q}_{\nu r}(\bar{a})) \int_{\mathcal{Q}_{\nu r}(\bar{a})} |a^{ij} - (a^{ij})_{\mathcal{Q}_{\nu r}(\bar{a})}| \mu(dtdx) \\ &\leq N(\nu r)^{\alpha+d+2} a_{2R, \kappa}^{\#(\theta)}. \end{aligned}$$

It follows that

$$J \leq N(a_{2R,\kappa}^{\#(\theta)})^{1/\beta'} \cdot \mathbb{M}^{1/\beta}(|u_{xx}|^{\beta q})(t, x).$$

Remember that $r/a \geq \delta_0 = \delta_0(\varepsilon)$. Thus for (5.7) it is enough to take ν so that $N(1 + \nu\delta_0)^{-q} \leq \varepsilon$ and observe that

$$\frac{N \cdot \nu^{d+1}}{r/a} (1 + \nu r/a)^{p+\theta-d+1} \leq N(\alpha) < \infty.$$

The lemma is proved. \square

Corollary 5.5. *Suppose the the assumptions in Lemma 5.4 are satisfied.*

(i) *The for any $\varepsilon > 0$ and $(t, x) \in \Omega$,*

$$(u_{xx})^\#(t, x) \leq \varepsilon \mathbb{M}^{1/q}(|u_{xx}|^q) + N \mathbb{M}^{1/q}(|f|^q)(t, x) + N(a_{2R,\kappa}^{\#(\theta)})^{1/(q\beta')} \cdot \mathbb{M}^{1/(q\beta)}(|u_{xx}|^{\beta q})(t, x),$$

where $N = N(\varepsilon, \theta, q, d, \delta, K)$ is independent of κ, t, x .

(ii)

$$\|Mu_{xx}\|_{\mathbb{L}_{p,\theta}(-\infty, \infty)}^p \leq N(d, p, \delta, K) \|Mf\|_{\mathbb{L}_{p,\theta}(-\infty, \infty)}^p + N(p) a_{2R,\kappa}^{\#(\theta)} \cdot \|Mu_{xx}\|_{\mathbb{L}_{p,\theta}(-\infty, \infty)}^p.$$

Proof. (i) is an easy consequence of Lemma 5.4 and Jensen's inequality. To prove (ii), take q and $\beta > 1$ so that $q < p$, $q\beta' = p$, and apply Theorems 2.3 and 2.4. \square

The following result is a parabolic version of Lemma 3.3 of [14]. Define $\mathbb{Q}(\kappa) := \cup_{R>0} \mathbb{Q}(R, \kappa)$.

Lemma 5.6. *For any $\varepsilon > 0$, there exist a constant $\kappa = \kappa(\varepsilon) \in (1/2, 1)$ and nonnegative functions $\eta_k \in C_0^\infty(\mathbb{R}_+^{d+1})$, $k = 1, 2, \dots$ so that (i) on \mathbb{R}_+^{d+1}*

$$\sum_k \eta_k^p \geq 1, \quad \sum_k \eta_k \leq N(d), \quad \sum_k (M|\eta_{kx}| + M^2|\eta_{kxx}| + M^2|\eta_{kt}|) \leq \varepsilon; \quad (5.15)$$

(ii) *for each k , $\text{supp } \eta_k \subset Q_k$ for some $Q_k \in \mathbb{Q}(\kappa)$.*

Proof. We modify the proof of Lemma 3.3 of [14]. Let

$$\mathbb{R}^{d-1} = \bigcup_{k=1}^{\infty} Q'_k, \quad \mathbb{R} = \bigcup_{\ell=1}^{\infty} I_\ell$$

be a decomposition of \mathbb{R}^{d-1} and \mathbb{R} into disjoint unit cubes Q'_k and I_ℓ respectively. Mollify the indicator function of each Q'_k and I_ℓ in such a way that thus obtained functions χ_k and $\hat{\chi}_\ell$ vanish outside of the twice dilated Q'_k and I_ℓ respectively (naturally, with center of dilation being that of Q'_k and I_ℓ respectively). Then (by multiplying by a large constant $c > 0$ to χ_k and $\hat{\chi}_\ell$ if necessary)

$$1 \leq \sum_k \chi_k^p \leq \left(\sum_k \chi_k\right)^p \leq N_0, \quad 1 \leq \sum_\ell \hat{\chi}_\ell^p \leq \left(\sum_\ell \hat{\chi}_\ell\right)^p \leq N_0$$

on \mathbb{R}^{d-1} and \mathbb{R} , respectively. Here the constant $N_0 \in (0, \infty)$ depends only on d and p . Furthermore, by Lemma 3.2 of [13], there exists a nonnegative function $\xi \in C_0^\infty(\mathbb{R}_+)$ such that assertion (i) of the present lemma holds on \mathbb{R}_+ with the collection $\{\xi(e^n x) : n \in \mathbb{Z}\}$ in place of $\{\eta_k(x) : k = 1, 2, \dots\}$.

We write $x = (x^1, x')$, fix a constant $r \in (0, 1)$ to be specified later, and introduce

$$\tau_k(x') = \chi_k(rx'), \quad \hat{\tau}_\ell(t) = \hat{\chi}_\ell(rt), \quad \eta_{nk\ell}(x) = \xi(e^n x^1) \tau_k(e^n x') \hat{\tau}_\ell(e^{2n} t).$$

Then

$$1 \leq \sum_{n,k,\ell} \eta_{nk\ell}^p \leq \left(\sum_{n,k,\ell} \eta_{nk\ell} \right)^p \leq N \quad \text{on } \mathbb{R}_+^{d+1} \quad (5.16)$$

with constant $N \in (0, \infty)$ depending only on d and p .

Now, for any multi-index $\alpha = (\alpha^1, \dots, \alpha^d)$ with $1 \leq |\alpha| \leq 2$, we have (with some constants $c_{\beta\gamma}$)

$$M^{|\alpha|} D_x^\alpha \eta_{nk\ell}(t, x) = (x^1)^{|\alpha|} e^{n|\alpha|} \sum_{\beta+\gamma=\alpha} c_{\beta\gamma} \xi^{(\beta_1)}(e^n x^1) (D^\gamma \tau_k)(e^n x') \hat{\tau}_\ell(e^{2n} t),$$

and

$$M^2(\eta_{nk\ell})_t = (x^1)^2 e^{2n} \xi(e^n x^1) \tau(e^n x') (\hat{\tau}_\ell)'(e^{2n} t).$$

Hence,

$$\sum_{n,k,\ell} |M^{|\alpha|} D_x^\alpha \eta_{nk\ell}(x)| \leq N_0 \sum_{\beta+\gamma=\alpha} c_{\beta\gamma} I_1(\gamma) I_2(\alpha, \beta),$$

where

$$I_1(\gamma) = \sup_{x'} \sum_k |D^\gamma \tau_k(x')| = r^{|\gamma|} \sup_{x'} \sum_k |D^\gamma \chi_k(x')|,$$

$$I_2(\alpha, \beta) = \sup_{x^1 \geq 0} \sum_n (x^1)^{|\alpha|} e^{n|\alpha|} |\xi^{(\beta_1)}(e^n x^1)| = \sup_{t \in \mathbb{R}} \sum_n e^{(n+t)|\alpha|} |\xi^{(\beta_1)}(e^{n+t})|.$$

Obviously I_1 is finite. That I_2 is also finite is seen from its representation as the supremum of a continuous 1-periodic function. Moreover, if $\gamma = 0$, then $c_{\beta\gamma} \neq 0$ only if $\beta_1 = |\alpha|$, in which case $c_{\beta\gamma} = 1$ and, by the construction of ξ , we have $I_2(\alpha, \beta) \leq \varepsilon$. It follows that

$$\sum_{n,k,\ell} |M^{|\alpha|} D_x^\alpha \eta_{nk\ell}(x)| \leq N(d)\varepsilon + N(\varepsilon, q, d)r. \quad (5.17)$$

Similar calculus shows

$$\sum_{n,k,\ell} |M^2(\eta_{nk\ell})_t| \leq N(\varepsilon, q, d)r. \quad (5.18)$$

We renumber the set $\{\eta_{nk\ell} : n = 0, \pm 1, \dots, k = 1, 2, \dots, \ell = 1, 2, \dots\}$ and write it as $\{\eta_k : k = 1, 2, \dots\}$. Then from (5.17) and (5.18) we see how to choose r in order to satisfy the last inequality in (5.15) with $N(d)\varepsilon$ in place of ε . This proves (i).

Now we prove (ii). Let $(\alpha, \beta) \subset \mathbb{R}_+$ so that $\text{supp } \xi \subset (\alpha, \beta)$. The above proofs show that $\text{supp } \eta_{0k\ell} \subset (t_{k\ell}, t_{k\ell} + r_0) \times (\alpha, \beta) \times B_r(x'_{k\ell}) =: Q_{0k\ell}$ for some $t_{k\ell}, x'_{k\ell}, r_0, r$ with r_0, r independent of k, ℓ . By increasing β and adjusting r_0, r if necessary we may assume that $Q_{0k\ell} \in \mathbb{Q}(\kappa)$ for some $\kappa \in (0, 1)$, independent of k, ℓ . Finally it is enough to note that

$$\text{supp } \eta_{nk\ell} \subset (e^{-2n} t_{k\ell}, e^{-2n} t_{k\ell} + e^{-2n} r_0) \times (e^{-2n} \alpha, e^{-2n} \beta) \times B_{e^{-2n} r}(e^{-2n} x'_{k\ell}) := Q_{nk\ell} \in \mathbb{Q}(\kappa).$$

The lemma is proved. \square

Lemma 5.7. Let $u \in C^\infty(\Omega)$ and denote $f = u_t + a^{ij}u_{x^i x^j}$.

(i) There exists a constant $\kappa_0 = \kappa_0(d, p, \theta, \delta, K) \in (0, 1)$ so that if $\kappa \in [\kappa_0, 1]$, then

$$\|Mu_{xx}\|_{\mathbb{L}_{p,\theta}^p(-\infty,\infty)}^p \leq N(d, p, \delta, K, \kappa) \left(\|Mf\|_{\mathbb{L}_{p,\theta}^p(-\infty,\infty)}^p + a_\kappa^{\#(\theta)} \|Mu_{xx}\|_{\mathbb{L}_{p,\theta}^p(-\infty,\infty)}^p \right). \quad (5.19)$$

(ii) If $u(t, x) = 0$ whenever $x^1 \geq R$, then

$$\|Mu_{xx}\|_{\mathbb{L}_{p,\theta}^p(-\infty,\infty)}^p \leq N(d, p, \delta, K) \left(\|Mf\|_{\mathbb{L}_{p,\theta}^p(-\infty,\infty)}^p + a_{R\kappa_0, \kappa_0}^{\#(\theta)} \|Mu_{xx}\|_{\mathbb{L}_{p,\theta}^p(-\infty,\infty)}^p \right),$$

where $R_{\kappa_0} := 2R(1 + \kappa_0)/(1 - \kappa_0)$.

Proof. (i) Fix $\varepsilon \in (0, 1)$ which will be specified later. Take $\{\eta_n : n = 1, 2, \dots\}$ from Lemma 5.6 corresponding to ε . Then since $\sum_n \eta_n^p \geq 1$,

$$\begin{aligned} \|Mu_{xx}\|_{\mathbb{L}_{p,\theta}^p(-\infty,\infty)}^p &\leq \sum_n \|\eta_n Mu_{xx}\|_{\mathbb{L}_{p,\theta}^p(-\infty,\infty)}^p \\ &\leq \sum_n \left(\|M(\eta_n u)_{xx}\|_{\mathbb{L}_{p,\theta}^p(-\infty,\infty)}^p + \|u_x M(\eta_n)_x\|_{\mathbb{L}_{p,\theta}^p(-\infty,\infty)}^p + \|M^{-1}u M^2(\eta_n)_{xx}\|_{\mathbb{L}_{p,\theta}^p(-\infty,\infty)}^p \right). \end{aligned}$$

Note that $u^n := u\eta_n$ satisfies

$$u_t^n + a^{ij}u_{x^i x^j}^n = f_n := u(\eta_n)_t + 2a^{ij}u_{x^i}(\eta_n)_{x^j} + a^{ij}u(\eta_n)_{x^i x^j} + f\eta_n,$$

and by Lemma 5.6 we have $\text{supp } u^n \subset Q_n \in \mathbb{Q}(\kappa)$ for some $\kappa = \kappa(\varepsilon) \in (0, 1)$. Then by Corollary 5.5,

$$\|Mu_{xx}^n\|_{\mathbb{L}_{p,\theta}^p(-\infty,\infty)}^p \leq N\|Mf_n\|_{\mathbb{L}_{p,\theta}^p(-\infty,\infty)}^p + N(p, q)a_\kappa^{\#(\theta)} \cdot \|Mu_{xx}^n\|_{\mathbb{L}_{p,\theta}^p(-\infty,\infty)}^p.$$

It follows that

$$\begin{aligned} \|Mu_{xx}\|_{\mathbb{L}_{p,\theta}^p(-\infty,\infty)}^p &\leq N\varepsilon^p (\|M^{-1}u\|_{\mathbb{L}_{p,\theta}^p(-\infty,\infty)}^p + \|u_x\|_{\mathbb{L}_{p,\theta}^p(-\infty,\infty)}^p) \\ &\quad + Na_\kappa^{\#(\theta)} \|Mu_{xx}\|_{\mathbb{L}_{p,\theta}^p(-\infty,\infty)}^p + \varepsilon^p a_\kappa^{\#(\theta)} \|M^{-1}u\|_{\mathbb{L}_{p,\theta}^p(-\infty,\infty)}^p \\ &\quad + \varepsilon^p a_\kappa^{\#(\theta)} \|u_x\|_{\mathbb{L}_{p,\theta}^p(-\infty,\infty)}^p + N\|Mf\|_{\mathbb{L}_{p,\theta}^p(-\infty,\infty)}^p. \end{aligned}$$

Since $\|M^{-1}u\|_{L_{p,\theta}} + \|u_x\|_{L_{p,\theta}} \leq N\|Mu_{xx}\|_{L_{p,\theta}}$, we get (i) if ε is sufficiently small.

(ii) Now let $\text{supp } \eta_n \subset Q_n = \mathcal{Q}_{\kappa_0}(t_0, x_0^1, x_0')$. Note that $u\eta_n = 0$ if $Q_n \notin \mathcal{Q}_{\frac{1+\kappa_0}{1-\kappa_0}R, \kappa_0}$. Thus in the proof of (i), we only need to consider the case $Q_n \in \mathcal{Q}_{\frac{1+\kappa_0}{1-\kappa_0}R, \kappa_0}$. Therefore (ii) follows from Corollary 5.5(ii) and the proof of (i). \square

6 L_p -theory on \mathbb{R}_+^d

Definition 6.1. Let $-\infty \leq S < T \leq \infty$. We write $u \in \mathfrak{H}_{p,\theta}^{\gamma+2}(S, T)$ if $u \in M\mathbb{H}_{p,\theta}^{\gamma+2}(S, T)$, $u(S, \cdot) \in U_{p,\theta}^{\gamma+2}(u(-\infty, \cdot) := 0 \text{ if } S = -\infty)$, and for some $\tilde{f} \in M^{-1}\mathbb{H}_{p,\theta}^\gamma(S, T)$ it holds that for any $\phi \in C_0^\infty(\mathbb{R}^d)$

$$(u(t, \cdot), \phi) = (u(S, \cdot), \phi) + \int_S^t (\tilde{f}(s, \cdot), \phi) ds, \quad t \in (S, T). \quad (6.1)$$

In this case we write $u_t = \tilde{f}$. The norm in $\mathfrak{H}_{p,\theta}^{\gamma+2}(S, T)$ is defined by

$$\|u\|_{\mathfrak{H}_{p,\theta}^{\gamma+2}(S, T)} = \|M^{-1}u\|_{\mathbb{H}_{p,\theta}^{\gamma+2}(S, T)} + \|Mu_t\|_{\mathbb{H}_{p,\theta}^{\gamma}(S, T)} + \|u(S, \cdot)\|_{U_{p,\theta}^{\gamma+2}}.$$

Define $\mathfrak{H}_{p,\theta}^{\gamma+2}(T) := \mathfrak{H}_{p,\theta}^{\gamma+2}(0, T)$, $\mathfrak{H}_{p,\theta}^{\gamma+2} := \mathfrak{H}_{p,\theta}^{\gamma+2}(0, \infty)$ and $\mathfrak{H}_{p,\theta,0}^{\gamma+2}(T) := \mathfrak{H}_{p,\theta}^{\gamma+2}(T) \cap \{u : u(0) = 0\}$.

Theorem 6.2. (i) The space $\mathfrak{H}_{p,\theta}^{\gamma+2}(S, T)$ is a Banach space.

(ii) If $T < \infty$, then for any $u \in \mathfrak{H}_{p,\theta,0}^{\gamma+2}(T)$,

$$\sup_{t \leq T} \|u(t)\|_{H_{p,\theta}^{\gamma+1}}^p \leq N(d, p, \theta, T) \|u\|_{\mathfrak{H}_{p,\theta}^{\gamma+2}(T)}^p.$$

In particular, for any $t \leq T$,

$$\|u\|_{\mathbb{H}_{p,\theta}^{\gamma+1}(T)}^p \leq \int_0^T \sup_{r \leq s} \|u(r)\|_{H_{p,\theta}^{\gamma+1}}^p ds \leq N \int_0^t \|u\|_{\mathfrak{H}_{p,\theta}^{\gamma+2}(s)}^p ds. \quad (6.2)$$

(iii) For any nonnegative integer $n \geq \gamma + 2$, the set

$$\mathfrak{H}_{p,\theta}^{\gamma+2}(T) \cap \bigcap_{k=1}^{\infty} C([0, T], C_0^n(G_k))$$

where $G_k = (1/k, k) \times \{|x'| < k\}$ is dense in $\mathfrak{H}_{p,\theta}^{\gamma+2}(T)$.

Proof. See Theorem 2.9 and Theorem 2.11 of [21]. Actually in [21], (i) is proved only for $p \geq 2$ based on Theorems 4.2 and 7.2 in [18]. But by inspecting the proofs of Theorems 4.2 and 7.2 in [18] one can easily check that in our (deterministic) case the result holds for all $p > 1$. \square

Remark 6.3. It is easy to check that any function $u \in \mathfrak{H}_{p,\theta}^2(-\infty, \infty)$ can be approximated by functions in $C_0^\infty(\Omega)$. Thus Lemma 5.7 holds for any $u \in \mathfrak{H}_{p,\theta}^2(-\infty, \infty)$.

Here are some interior Hölder estimates of functions in the space $\mathfrak{H}_{p,\theta}^{\gamma+2}(T)$.

Theorem 6.4. Let $p > 2$ and assume

$$2/p < \alpha < \beta \leq 1, \quad \gamma + 2 - \beta - d/p = k + \varepsilon,$$

where $k \in \{0, 1, 2, \dots\}$ and $\varepsilon \in (0, 1]$. Denote $\delta = \beta - 1 + \theta/p$. Then for any $u \in \mathfrak{H}_{p,\theta}^{\gamma+2}(T)$ and multi-indices i, j such that $|i| \leq j$ and $|j| = k$,

(i) the functions $D^i u(t, x)$ are continuous in $[0, T] \times \mathbb{R}_+^d$ and

$$M^{\delta+|i|} D^i u(t, \cdot) - M^{\delta+|i|} D^i u(0, \cdot) \in C^{\alpha/2-1/p}([0, T], C(\mathbb{R}_+^d));$$

(ii) there exists a constant $N = N(p, d, \alpha, \beta)$ so that

$$\sup_{t, s \leq T} \left(\frac{|M^{\delta+|i|} D^i(u(t) - u(s))|_{C(\mathbb{R}_+^d)}}{|t - s|^{\alpha/2-1/p}} + \frac{[M^{\delta+|j|+\varepsilon} D^j(u(t) - u(s))]_{C^\varepsilon}}{|t - s|^{\alpha/2-1/p}} \right) \leq N T^{(\beta-\alpha)/2} \|u\|_{\mathfrak{H}_{p,\theta}^{\gamma+2}(T)}. \quad (6.3)$$

Proof. See Theorem 4.7 of [15]. □

Throughout this section we assume the following.

Assumption 6.5. There exist constants $\delta, K > 0$ so that

$$\delta|\xi|^2 \leq a^{ij}(t, x)\xi^i\xi^j \leq K|\xi|^2, \quad \forall \xi \in \mathbb{R}^d. \quad (6.4)$$

Theorem 6.6. Let $p \in (1, \infty)$, $\theta \in (d-1, d-1+p)$ and $T \in (0, \infty]$. Take $\kappa_0 \in (0, 1)$ from Lemma 5.7. Assume that there exists a constant $\beta > 0$ so that

$$|x^1 b^i| + |(x^1)^2 c| \leq \beta, \quad \forall t, x. \quad (6.5)$$

(i) Then there exists constants $\varepsilon_0, \beta_0 > 0$ depending only on d, p, θ, δ and K so that if $a_{\kappa_0}^{\#(\theta)} < \varepsilon_0$ and $\beta \leq \beta_0$ then for any $f \in M^{-1}\mathbb{L}_{p,\theta}(T)$ and $u_0 \in U_{p,\theta}^2$ the equation

$$u_t = a^{ij}u_{x^i x^j} + b^i u_{x^i} + cu + f, \quad u(0) = u_0 \quad (6.6)$$

admits a unique solution $u \in \mathfrak{H}_{p,\theta}^2(T)$, and for this solution we have

$$\|u\|_{\mathfrak{H}_{p,\theta}^2(T)} \leq N \left(\|Mf\|_{\mathbb{L}_{p,\theta}(T)} + \|u_0\|_{U_{p,\theta}^2} \right), \quad (6.7)$$

where $N = N(p, \theta, \delta_0, K)$.

(ii) Let $u \in \mathfrak{H}_{p,\theta}^2(T)$ be a solution of equation (7.9) and $u(t, x) = 0$ whenever $x^1 \geq R$. Then the estimate (6.7) holds true if $a_{R_{\kappa_0}, \kappa_0}^{\#(\theta)} < \varepsilon_0$, where $R_{\kappa_0} := 2R(1 + \kappa_0)/(1 - \kappa_0)$.

Remark 6.7. It is known (see Remark 3.6 of [21]) that if $\theta \notin (d-1, d-1+p)$, then Theorem 6.6 is not true even for the heat equation $u_t = \Delta u + f$.

Proof of Theorem 6.6. As usual, we assume $u_0 = 0$ (see the proof of Theorem 5.1 in [18]). Take $N = N(d, p, \theta, \delta, K, \kappa_0)$ from (5.19) and assume that $a_{\kappa_0}^{\#(\theta)} < \varepsilon_0 := 1/(2N)$.

Case 1. Let $T = \infty$ and $b^i = c = 0$. Due to Lemma 3.2 and the method of continuity, we only prove that estimate (6.7) holds given that a solution $u \in \mathfrak{H}_{p,\theta}^2(T)$ already exists.

Define $v(t, x) = u(t, x)I_{t>0}$ and $\bar{f} = fI_{t>0}$, then $v \in M^{-1}\mathbb{H}_{p,\theta}^2(-\infty, \infty)$ and v satisfies (see (6.1))

$$v_t = a^{ij}u_{x^i x^j} + \bar{f}, \quad (t, x) \in \mathbb{R}_+^{d+1}.$$

By Lemma 5.7 and Remark 6.3,

$$\|Mv_{xx}\|_{\mathbb{L}_{p,\theta}(-\infty, \infty)} \leq N\|Mf\|_{\mathbb{L}_{p,\theta}(\infty)}.$$

This certainly proves (7.10).

Case 2. Let $T < \infty$ and $b^i = c = 0$. The existence of solutions in $\mathfrak{H}_{p,\theta}^{\gamma+2}(T)$ is an easy consequence of Case 1. Now suppose that $u \in \mathfrak{H}_{p,\theta}^{\gamma+2}(T)$ is a solution of (7.9). By the result of Case 1, the equation

$$v_t = \Delta v + (a^{ij}u_{x^i x^j} + f - \Delta u)I_{t \leq T}, \quad t > 0; \quad v(0, \cdot) = 0 \quad (6.8)$$

has a unique solution $v \in \mathfrak{H}_{p,\theta}^{\gamma+2}(0, \infty)$. Then $v - u$ satisfies

$$(v - u)_t = \Delta(v - u), \quad t \in (0, T); \quad (v - u)(0, \cdot) = 0.$$

It follows from Lemma 3.2 that $u = v$ for $t \in [0, T]$. For $t \geq 0$, define

$$a_T^{ij} = a^{ij} I_{t \leq T} + \delta^{ij} I_{t > T}.$$

Then (6.8) and the fact $u = v$ for $t \in [0, T]$ show that v satisfies (replace u by v for $t \leq T$ in (6.8))

$$v_t = a_T^{ij} v_{x^i x^j} + f I_{t < T}, \quad t > 0; \quad v(0, \cdot) = 0. \quad (6.9)$$

By Case 1, $v \in \mathfrak{H}_{p,\theta}^{\gamma+2}(\infty)$ is the unique solution of (6.9), and $u = v$ on $[0, T]$ whenever u is a solution of (7.9) on $[0, T]$. This obviously yields the uniqueness.

Case 3. General case. Again we only prove that there exists β_0 so that if $a_{\kappa_0}^{\#(\theta)} < \varepsilon_0$ and $\beta \leq \beta_0$ then estimate (6.7) holds given that a solution $u \in \mathfrak{H}_{p,\theta}^2(T)$ already exists. Obviously by the results of Case 1 and 2,

$$\begin{aligned} \|M^{-1}u\|_{\mathbb{H}_{p,\theta}^2(T)} &\leq N \|M(b^i u_{x^i} + cu + f)\|_{\mathbb{L}_{p,\theta}(T)} \\ &\leq N \sup |x^1 b^i| \|u_x\|_{\mathbb{L}_{p,\theta}(T)} + N \sup |(x^1)^2 M^{-1}u|_{\mathbb{L}_{p,\theta}(T)} + N \|Mf\|_{\mathbb{L}_{p,\theta}(T)} \\ &\leq N\beta \|M^{-1}u\|_{\mathbb{H}_{p,\theta}^2(T)} + N \|Mf\|_{\mathbb{L}_{p,\theta}(T)}. \end{aligned}$$

Thus it is enough to take β_0 so that $N\beta_0 \leq 1/2$. The theorem is proved. \square

7 L_p -theory on bounded C^1 domains

Assumption 7.1. The domain \mathcal{O} is of class C_u^1 . In other words, there exist constants $r_0, K_0 \in (0, \infty)$ so that for any $x_0 \in \partial\mathcal{O}$ there exists a one-to-one continuously differentiable mapping Ψ of $B_{r_0}(x_0)$ onto a domain $J \subset \mathbb{R}^d$ such that

- (i) $J_+ := \Psi(B_{r_0}(x_0) \cap \mathcal{O}) \subset \mathbb{R}_+^d$ and $\Psi(x_0) = 0$;
- (ii) $\Psi(B_{r_0}(x_0) \cap \partial\mathcal{O}) = J \cap \{y \in \mathbb{R}^d : y^1 = 0\}$;
- (iii) $\|\Psi\|_{C^1(B_{r_0}(x_0))} \leq K_0$ and $|\Psi^{-1}(y_1) - \Psi^{-1}(y_2)| \leq K_0|y_1 - y_2|$ for any $y_i \in J$;
- (iv) Ψ_x is uniformly continuous in for $B_{r_0}(x_0)$.

To proceed further we introduce some well known results from [8] and [14] (see also [23] for the details). Denote $\rho(x) := \text{dist}(x, \partial\mathcal{O})$.

Lemma 7.2. *Let the domain \mathcal{O} be of class C_u^1 . Then*

- (i) *there is a bounded real-valued function ψ defined in $\bar{\mathcal{O}}$ such that the functions $\psi(x)$ and $\rho(x)$ are comparable. In other words, $N^{-1}\rho(x) \leq \psi(x) \leq N\rho(x)$ with some constant N independent of x ,*
- (ii) *for any multi-index α ,*

$$\sup_{\mathcal{O}} \psi^{|\alpha|}(x) |D^\alpha \psi_x(x)| < \infty. \quad (7.1)$$

First we introduce Banach spaces $H_{p,\theta}^\gamma(\mathcal{O})$, which correspond to the spaces $H_{p,\theta}^\gamma$ on \mathbb{R}_+^d . Take $\zeta \in C_0^\infty(\mathbb{R}_+)$ satisfying (2.5), which is

$$\sum_{n=-\infty}^{\infty} \zeta(e^{n+x}) > c > 0, \quad \forall x \in \mathbb{R}.$$

For $x \in \mathcal{O}$ and $n \in \mathbb{Z} = \{0, \pm 1, \dots\}$ define

$$\zeta_n(x) = \zeta(e^n \psi(x)).$$

Then we have $\sum_n \zeta_n \geq c$ in \mathcal{O} and

$$\zeta_n \in C_0^\infty(\mathcal{O}), \quad |D^m \zeta_n(x)| \leq N(m) e^{mn}.$$

For $\theta, \gamma \in \mathbb{R}$, let $H_{p,\theta}^\gamma(\mathcal{O})$ be the set of all distributions u on \mathcal{O} such that

$$\|u\|_{H_{p,\theta}^\gamma(\mathcal{O})}^p := \sum_{n \in \mathbb{Z}} e^{n\theta} \|\zeta_{-n}(e^n \cdot) u(e^n \cdot)\|_{H_p^\gamma}^p < \infty. \quad (7.2)$$

It is known (see, for instance, [26] or [14]) that up to equivalent norms the space $H_{p,\theta}^\gamma(\mathcal{O})$ is independent of the choice of ζ and ψ . Moreover if $\gamma = n$ is a non-negative integer then

$$\|u\|_{H_{p,\theta}^\gamma(\mathcal{O})}^p \sim \sum_{|\alpha| \leq n} \int_{\mathcal{O}} |\rho^{|\alpha|} D^\alpha u(x)|^p \rho^{\theta-d}(x) dx. \quad (7.3)$$

Recall that if $\gamma = n$, then the space $H_{p,\theta}^\gamma$ is the collection of functions u on \mathbb{R}_+^d so that

$$\sum_{|\alpha| \leq n} \int_{\mathbb{R}_+^d} |(x^1)^{|\alpha|} D^\alpha u(x)|^p (x^1)^{\theta-d}(x) dx < \infty.$$

Denote $\psi(x, y) = \psi(x) \wedge \psi(y)$. For $n \in \mathbb{Z}$, $\mu \in (0, 1]$ and $k = 0, 1, 2, \dots$, define

$$\begin{aligned} |u|_C &= \sup_{\mathcal{O}} |u(x)|, \quad [u]_{C^\mu} = \sup_{x \neq y} \frac{|u(x) - u(y)|}{|x - y|^\mu}. \\ [u]_k^{(n)} &= [u]_{k,\mathcal{O}}^{(n)} = \sup_{\substack{x \in \mathcal{O} \\ |\beta|=k}} \psi^{k+n}(x) |D^\beta u(x)|, \end{aligned} \quad (7.4)$$

$$[u]_{k+\mu}^{(n)} = [u]_{k+\mu,\mathcal{O}}^{(n)} = \sup_{\substack{x,y \in \mathcal{O} \\ |\beta|=k}} \psi^{k+\mu+n}(x,y) \frac{|D^\beta u(x) - D^\beta u(y)|}{|x - y|^\mu}, \quad (7.5)$$

$$|u|_k^{(n)} = |u|_{k,\mathcal{O}}^{(n)} = \sum_{j=0}^k [u]_{j,\mathcal{O}}^{(n)}, \quad |u|_{k+\mu}^{(n)} = |u|_{k+\mu,\mathcal{O}}^{(n)} = |u|_{k,\mathcal{O}}^{(n)} + [u]_{k+\mu,\mathcal{O}}^{(n)}.$$

Below we collect some other properties of spaces $H_{p,\theta}^\gamma(\mathcal{O})$ taken from [26] (also see [14]).

Lemma 7.3. *Let $d-1 < \theta < d-1+p$.*

(i) *Assume that $\gamma - d/p = m + \nu$ for some $m = 0, 1, \dots$ and $\nu \in (0, 1]$. Then for any $u \in H_{p,\theta}^\gamma(\mathcal{O})$ and $i \in \{0, 1, \dots, m\}$, we have*

$$|\psi^{i+\theta/p} D^i u|_C + [\psi^{m+\nu+\theta/p} D^m u]_{C^\nu} \leq c \|u\|_{H_{p,\theta}^\gamma(\mathcal{O})}.$$

(ii) Let $\alpha \in \mathbb{R}$, then $\psi^\alpha H_{p,\theta+\alpha p}^\gamma(\mathcal{O}) = H_{p,\theta}^\gamma(\mathcal{O})$,

$$\|u\|_{H_{p,\theta}^\gamma(\mathcal{O})} \leq c \|\psi^{-\alpha} u\|_{H_{p,\theta+\alpha p}^\gamma(\mathcal{O})} \leq c \|u\|_{H_{p,\theta}^\gamma(\mathcal{O})}.$$

(iii) There is a constant $c = c(d, p, \gamma, \theta)$ so that

$$\|af\|_{H_{p,\theta}^\gamma(\mathcal{O})} \leq c |a|_{|\gamma|_+}^{(0)} \|f\|_{H_{p,\theta}^\gamma(\mathcal{O})}.$$

(iv) $\psi D, D\psi : H_{p,\theta}^\gamma(\mathcal{O}) \rightarrow H_{p,\theta}^{\gamma-1}(\mathcal{O})$ are bounded linear operators, and

$$\|u\|_{H_{p,\theta}^\gamma(\mathcal{O})} \leq c \|u\|_{H_{p,\theta}^{\gamma-1}(\mathcal{O})} + c \|\psi Du\|_{H_{p,\theta}^{\gamma-1}(\mathcal{O})} \leq c \|u\|_{H_{p,\theta}^\gamma(\mathcal{O})},$$

$$\|u\|_{H_{p,\theta}^\gamma(\mathcal{O})} \leq c \|u\|_{H_{p,\theta}^{\gamma-1}(\mathcal{O})} + c \|D\psi u\|_{H_{p,\theta}^{\gamma-1}(\mathcal{O})} \leq c \|u\|_{H_{p,\theta}^\gamma(\mathcal{O})}.$$

Denote

$$\mathbb{H}_{p,\theta}^\gamma(\mathcal{O}, T) = L_p([0, T], H_{p,\theta}^\gamma(\mathcal{O})), \quad \mathbb{L}_{p,\theta}(\mathcal{O}, T) = \mathbb{H}_{p,\theta}^0(\mathcal{O}, T)$$

$$U_{p,\theta}^\gamma(\mathcal{O}) = \psi^{1-2/p} H_{p,\theta}^{\gamma-2/p}(\mathcal{O}).$$

Definition 7.4. We write $u \in \mathfrak{H}_{p,\theta}^{\gamma+2}(\mathcal{O}, T)$ if $u \in \psi \mathbb{H}_{p,\theta}^{\gamma+2}(\mathcal{O}, T)$, $u(0, \cdot) \in U_{p,\theta}^{\gamma+2}(\mathcal{O})$ and for some $f \in \psi^{-1} \mathbb{H}_{p,\theta}^\gamma(\mathcal{O}, T)$, it holds that $u_t = f$ in the sense of distributions, that is for any $\phi \in C_0^\infty(\mathcal{O})$, the equality

$$(u(t), \phi) = (u(0), \phi) + \int_0^t (f(s), \phi) ds$$

holds for all $t \leq T$. The norm in $\mathfrak{H}_{p,\theta}^{\gamma+2}(\mathcal{O}, T)$ is introduced by

$$\|u\|_{\mathfrak{H}_{p,\theta}^{\gamma+2}(\mathcal{O}, T)} = \|\psi^{-1} u\|_{\mathbb{H}_{p,\theta}^{\gamma+2}(\mathcal{O}, T)} + \|\psi u_t\|_{\mathbb{H}_{p,\theta}^\gamma(\mathcal{O}, T)} + \|u(0, \cdot)\|_{U_{p,\theta}^{\gamma+2}(\mathcal{O})}.$$

Denote $\mathfrak{H}_{p,\theta,0}^{\gamma+2}(\mathcal{O}, T) = \mathfrak{H}_{p,\theta}^{\gamma+2}(\mathcal{O}, T) \cap \{u : u(0) = 0\}$.

Lemma 7.5. There exists a constant $N = N(d, p, \theta, \gamma, T)$ such that for any $u \in \mathfrak{H}_{p,\theta,0}^{\gamma+2}(T)$,

$$\sup_{t \leq T} \|u(t)\|_{H_{p,\theta}^{\gamma+1}(\mathcal{O})} \leq N \|u\|_{\mathfrak{H}_{p,\theta}^{\gamma+2}(\mathcal{O}, T)}.$$

In particular, for any $t \leq T$,

$$\|u\|_{\mathbb{H}_{p,\theta}^{\gamma+1}(\mathcal{O}, t)}^p \leq N \int_0^t \|u\|_{\mathfrak{H}_{p,\theta}^{\gamma+2}(\mathcal{O}, s)}^p ds.$$

Proof. See inequality (2.21) of [27]. Actually there is a restriction $p \geq 2$ in (2.21) of [27], but by inspecting the proofs of Theorems 4.2 and Theorem 7.1 in [18] one can easily check that in our (deterministic) case the result holds for all $p > 1$. \square

Denote $B_r(x) := \{y \in \mathbb{R}^d : |x - y| < r\}$ and $Q_r(t, x) := (t, t + r^2) \times B_r(x)$. As before, we define weighted mean oscillation on $Q_r(x)$ with respect to measure $\nu(dx) = \rho^{\theta-d+p} dx$

$$\begin{aligned} \text{osc}_x^\theta(a, Q_r(t, x)) &= r^{-2} \int_t^{t+r^2} \left(\int_{B_r(x)} \int_{B_r(x)} |a(s, y) - a(s, z)| \nu(dy) \nu(dz) \right) ds \\ &= \frac{1}{r^2 (\nu(B_r(x)))^2} \int_t^{t+r^2} \left(\int_{B_r(x)} \int_{B_r(x)} |a(s, y) - a(s, z)| \nu(dy) \nu(dz) \right) ds. \end{aligned}$$

Denote $\mathcal{O}_R := \{x \in \mathcal{O} : \rho(x) > R\}$ and $\mathcal{O}_R^c := \mathcal{O} \setminus \mathcal{O}_R$. For $\kappa \in (0, 1]$ and $R > 0$, let $\mathbf{Q}(R, \kappa)$ be the collection of all $Q_r(t, x)$ so that $r \leq \kappa\rho(x)$ and $Q_r(t, x) \subset \mathbb{R} \times \mathcal{O}_R^c$. Define

$$a_{R, \kappa}^{\#(\theta)} = a_{R, \kappa, \mathcal{O}}^{\#(\theta)} = \sup_{\mathcal{Q} \in \mathbf{Q}(R, \kappa)} \text{osc}_x^\theta(a, \mathcal{Q}), \quad a_\kappa^{\#(\theta)} = a_{\kappa, \mathcal{O}}^{\#(\theta)} = \sup_{R > 0} a_{R, \kappa}^{\#(\theta)}.$$

Recall that

$$\text{osc}_x(a, Q_r(t, x)) = \frac{1}{r^2 |B_r(x)|^2} \int_t^{t+r^2} \left(\int_{B_r(x)} \int_{B_r(x)} |a(s, y) - a(s, z)| dy dz \right) ds.$$

For a subset $\mathcal{U} \subset \mathcal{O}$ we say that $a = (a^{ij})$ is VMO in \mathcal{U} if

$$\lim_{r \rightarrow 0} \sup_{Q_r(t, x) \cap \mathcal{U} \neq \emptyset} \text{osc}_x(a, Q_r(t, x)) = 0.$$

Throughout this section we assume the following.

Assumption 7.6. There exist constants $\delta, K > 0$ so that

$$\delta |\xi|^2 \leq a^{ij}(t, x) \xi^i \xi^j \leq K |\xi|^2, \quad \forall \xi \in \mathbb{R}^d.$$

Here is the main result of this article.

Theorem 7.7. *Assume*

$$a = (a^{ij}) \text{ is VMO in } \mathcal{O}_\varepsilon \quad \text{for any } \varepsilon > 0 \tag{7.6}$$

$$\lim_{\rho(x) \rightarrow 0} \sup_t (\rho(x) |b^i(t, x)| + \rho^2(x) |c(t, x)|) = 0 \tag{7.7}$$

Then there exist constants $\varepsilon_1, \kappa_1 \in (0, 1)$ so that if

$$\lim_{R \rightarrow 0} a_{R, \kappa_1}^{\#(\theta)} < \varepsilon_1, \tag{7.8}$$

then for any $f \in \psi^{-1} \mathbb{L}_{p, \theta}(\mathcal{O}, T)$ and $u_0 \in U_{p, \theta}^2(\mathcal{O})$ the equation

$$u_t = a^{ij} u_{x^i x^j} + b^i u_{x^i} + cu + f, \quad u(0) = u_0 \tag{7.9}$$

admits a unique solution $u \in \mathfrak{H}_{p, \theta}^2(\mathcal{O}, T)$, and for this solution we have

$$\|u\|_{\mathfrak{H}_{p, \theta}^2(\mathcal{O}, T)} \leq N \left(\|\psi f\|_{\mathbb{L}_{p, \theta}(\mathcal{O}, T)} + \|u_0\|_{U_{p, \theta}^2(\mathcal{O})} \right), \tag{7.10}$$

where $N = N(p, \theta, \delta_0, K, T)$.

Remark 7.8. (i) By inspecting our proof, one easily checks that (7.8) and (7.6) can be replaced by

$$a_{R, \kappa_1}^{\#(\theta)} < \varepsilon_1 \quad \text{for some } R > 0, \quad \text{and} \quad a = (a^{ij}) \text{ is VMO in } \mathcal{O}_R.$$

(ii) Obviously, (7.8) and (7.6) are certainly satisfied if a is VMO in \mathcal{O} (see Remark 5.2).

(iii) Our proof shows that (7.7) can be replaced by

$$\lim_{\rho(x) \rightarrow 0} \sup_t (\rho(x) |b^i(t, x)| + \rho^2(x) |c(t, x)|) < \beta$$

for some $\beta > 0$.

Proof of Theorem 7.7

See Theorem 2.10 of [14] for the case $a^{ij} = \delta^{ij}$, $b^i = c = 0$. Hence, due to the method of continuity, we only need to show that (7.10) holds given that a solution $u \in \mathfrak{H}_{p,\theta}^2(T)$ already exists. Let $u \in \mathfrak{H}_{p,\theta}^2(\mathcal{O}, T)$ be a solution of equation (7.9). By Theorem 2.10 of [14], the equation

$$v_t = \Delta v, \quad v(0) = u_0$$

has a unique solution $v \in \mathfrak{H}_{p,\theta}^2(\mathcal{O}, T)$, and furthermore

$$\|v\|_{\mathfrak{H}_{p,\theta}^2(\mathcal{O}, T)} \leq N \|u_0\|_{U_{p,\theta}^2(\mathcal{O})}.$$

Thus considering $u - v$, we assume $u_0 = 0$.

Let $x_0 \in \partial\mathcal{O}$ and Ψ be a function from Assumption 7.1. In [14] it is shown that Ψ can be chosen in such a way that for any non-negative integer n

$$|\Psi_x|_{n, B_{r_0}(x_0) \cap \mathcal{O}}^{(0)} + |\Psi_x^{-1}|_{n, J_+}^{(0)} < N(n) < \infty \quad (7.11)$$

and

$$\rho(x) \Psi_{xx}(x) \rightarrow 0 \quad \text{as } x \in B_{r_0}(x_0) \cap \mathcal{O}, \text{ and } \rho(x) \rightarrow 0, \quad (7.12)$$

where the constants $N(n)$ and the convergence in (7.12) are independent of x_0 . Define $r = r_0/K_0$ and fix smooth functions $\eta \in C_0^\infty(B_{r/2}(0))$ such that $0 \leq \eta \leq 1$, and $\eta = 1$ in $B_{r/4}(0)$. Observe that $\Psi(B_{r_0}(x_0))$ contains B_r and $a^{ij}(\Psi^{-1}(x))$ is well defined for any $x \in B_r(0)$. For $t > 0$, $x \in \mathbb{R}_+^d$ let us introduce

$$\hat{a}^{ij}(t, x) := \eta(x) \left(\sum_{l,m=1}^d a^{lm}(t, \Psi^{-1}(x)) \partial_l \Psi^i(\Psi^{-1}(x)) \partial_m \Psi^j(\Psi^{-1}(x)) \right) + \delta^{ij}(1 - \eta(x)),$$

$$\hat{b}^i(t, x) := \eta(x) \left[\sum_{l,m} a^{lm}(t, \Psi^{-1}(x)) \cdot \partial_{lm} \Psi^i(\Psi^{-1}(x)) + \sum_l b^l(t, \Psi^{-1}(x)) \cdot \partial_l \Psi^i(\Psi^{-1}(x)) \right],$$

$$\hat{c}(t, x) := \eta(x) c(t, \Psi^{-1}(x)).$$

Then by (7.7) and (7.12) one can easily find $r_1 > 0$ satisfying

$$\sup_{x^1 \leq K_0 r_1} \left(|x^1 \hat{b}^i| + (x^1)^2 \hat{c} \right) \leq \beta_0/2.$$

Denote

$$\bar{b}^i = \hat{b}^i I_{x^1 \leq K_0 r_1}, \quad \bar{c} = \hat{c} I_{x^1 \leq K_0 r_1}.$$

It is not hard to check that there exists $\kappa_1 \in (0, 1)$ so that (if R is sufficiently small)

$$\hat{a}_{R, \kappa_0, \mathbb{R}_+^d}^{\#(\theta)} \leq N a_{K_0 R, \kappa_1, \mathcal{O}}^{\#(\theta)} + N \eta_{R, \kappa_0, \mathbb{R}_+^d}^{\#(\theta)} + c(R),$$

where $N = N(K_0, \eta)$ and $c(R) \downarrow 0$ as $R \rightarrow 0$. Take ε_1 so that $N\varepsilon_1 \leq \varepsilon_0/2$. The if $\lim_{R \rightarrow 0} a_{R, \kappa_1, \mathcal{O}}^{\#(\theta)} \leq \varepsilon_1$ then for any sufficiently small R , we have $\hat{a}_{R_0, \kappa_0, \mathbb{R}_+^d}^{\#(\theta)} \leq \varepsilon_0$, where $R_0 = 2R(1 + \kappa_0)/(1 - \kappa_0)$.

Denote $\bar{r} = r/(4K_0) \wedge r_1 \wedge R_0/K_0$. Let ζ be a smooth function with support in $B_{\bar{r}}(x_0)$ and denote $v := (u\zeta)(\Psi^{-1})$ and continue v as zero in $\mathbb{R}_+^d \setminus \Psi(B_{\bar{r}}(x_0))$. Since $\eta = 1$ on $\Psi(B_{\bar{r}}(x_0))$, the function v satisfies

$$v_t = \hat{a}^{ij} v_{x^i x^j} + \bar{b}^i v_{x^i} + \bar{c}v + \hat{f}$$

where

$$\hat{f} = \tilde{f}^k(\Psi^{-1}), \quad \tilde{f}^k = -2a^{ij} u_{x^i} \zeta_{x^j} - a^{ij} u \zeta_{x^i x^j} - b^i u^r \zeta_{x^i} + \zeta f.$$

Next we observe that by Lemma 7.2 and Theorem 3.2 in [26] (or see [14]) for any $\nu, \alpha \in \mathbb{R}$ and $h \in \psi^{-\alpha} H_{p,\theta}^\nu(\mathcal{O})$ with support in $B_{\bar{r}}(x_0)$

$$\|\psi^\alpha h\|_{H_{p,\theta}^\nu(\mathcal{O})} \sim \|M^\alpha h(\Psi^{-1})\|_{H_{p,\theta}^\nu}. \quad (7.13)$$

Therefore we conclude that $v \in \mathfrak{H}_{p,\theta}^2(T)$, and by Theorem 6.6(ii) we have, for any $t \leq T$,

$$\|M^{-1}v\|_{\mathbb{H}_{p,\theta}^2(t)} \leq N \|M\hat{f}\|_{\mathbb{L}_{p,\theta}(t)}.$$

By using (7.13) again we obtain

$$\begin{aligned} \|\psi^{-1}u\zeta\|_{\mathbb{H}_{p,\theta}^2(\mathcal{O},t)} &\leq N \|a\zeta_x \psi u_x\|_{\mathbb{L}_{p,\theta}(\mathcal{O},t)} + N \|a\zeta_{xx} \psi u\|_{\mathbb{L}_{p,\theta}(\mathcal{O},t)} \\ &\quad + N \|\zeta_x \psi b u\|_{\mathbb{L}_{p,\theta}(\mathcal{O},t)} + N \|\zeta \psi f\|_{\mathbb{L}_{p,\theta}(\mathcal{O},t)}. \end{aligned}$$

Next, we easily check that

$$\sup_{t,x} (|\zeta_x a| + |\zeta_{xx} \psi a| + |\zeta_x \psi b|) < \infty$$

and conclude

$$\|\psi^{-1}u\zeta\|_{\mathbb{H}_{p,\theta}^2(\mathcal{O},t)} \leq N \|\psi u_x\|_{\mathbb{L}_{p,\theta}(\mathcal{O},t)} + N \|u\|_{\mathbb{L}_{p,\theta}(\mathcal{O},t)} + N \|\psi f\|_{\mathbb{L}_{p,\theta}(\mathcal{O},t)}.$$

Finally, to estimate the norm $\|\psi^{-1}u\|_{\mathbb{H}_{p,\theta}^2(\mathcal{O},t)}$, we introduce a partition of unity $\zeta_{(i)}, i = 0, 1, 2, \dots, M$ such that $\zeta_{(0)} \in C_0^\infty(\mathcal{O})$ and $\zeta_{(i)} \in C_0^\infty(B_{\bar{r}}(x_i))$, $x_i \in \partial\mathcal{O}$ for $i \geq 1$. Observe that since $u\zeta_{(0)}$ has compact support in \mathcal{O} , we get

$$\|\psi^{-1}u\zeta_{(0)}\|_{\mathbb{H}_{p,\theta}^2(\mathcal{O},t)} \sim \|u\zeta_{(0)}\|_{\mathbb{H}_p^2(t)}.$$

Thus we can estimate $\|\psi^{-1}u\zeta_{(0)}\|_{\mathbb{H}_{p,\theta}^2(\mathcal{O},t)}$ using Theorem 2.1 in [17] and the other norms as above. By summing up those estimates we get

$$\|\psi^{-1}u\|_{\mathbb{H}_{p,\theta}^2(\mathcal{O},t)} \leq N \|\psi u_x\|_{\mathbb{L}_{p,\theta}(\mathcal{O},t)} + N \|u\|_{\mathbb{L}_{p,\theta}(\mathcal{O},t)} + N \|\psi f\|_{\mathbb{L}_{p,\theta}(\mathcal{O},t)}.$$

Furthermore, we know (see Lemma 7.3) that

$$\|\psi u_x\|_{H_{p,\theta}^\gamma(\mathcal{O})} \leq N \|u\|_{H_{p,\theta}^{\gamma+1}(\mathcal{O})}.$$

Therefore it follows

$$\begin{aligned} \|u\|_{\mathfrak{H}_{p,\theta}^2(\mathcal{O},t)}^p &\leq N\|u\|_{\mathbb{H}_{p,\theta}^1(\mathcal{O},t)}^p + N\|\psi f\|_{\mathbb{L}_{p,\theta}(\mathcal{O},t)}^p \\ &\leq N\int_0^t \|u\|_{\mathfrak{H}_{p,\theta}^2(\mathcal{O},s)}^p ds + N\|\psi f\|_{\mathbb{L}_{p,\theta}(\mathcal{O},t)}^p, \end{aligned}$$

where Lemma 7.5 is used for the second inequality. Now (7.10) follows from Gronwall's inequality. The theorem is proved. \square

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